MULTI-POINT SESHADRI CONSTANTS ON RULED SURFACES

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Abstract. Let X be a surface and let L be an ample line bundle on X. We first obtain a lower bound for the Seshadri constant $\varepsilon(X, L, r)$, when $r \geq 2$. We then assume that X is a ruled surface and study Seshadri constants on X in greater detail. We also make precise computations of Seshadri constants on X in some cases.

1. Introduction

Seshadri constants have been defined by Demailly [7] as a measure of local positivity of a line bundle on a projective variety. The foundational idea is a criterion for ampleness given by Seshadri [16, Theorem 7.1]. For a detailed introduction and the state of current research on Seshadri constants, see [4].

Let X be a smooth complex projective variety and let L be a nef line bundle on X. Let $r \geq 1$ be an integer and let $x_1, \ldots, x_r$ be distinct points of X. The *Seshadri constant* of L at $x_1, \ldots, x_r \in X$ is defined as:

$$\varepsilon(X, L, x_1, \ldots, x_r) := \inf_{\substack{C \subset X \text{ a curve with} \\ C \cap \{x_1, \ldots, x_r\} \neq \emptyset}} \frac{L \cdot C}{\sum_{i=1}^{r} \text{mult}_{x_i} C}.$$

It is easy to see that the infimum above is the same as the infimum taken over irreducible, reduced curves C such that $C \cap \{x_1, \ldots, x_r\} \neq \emptyset$.

The Seshadri criterion for ampleness says that a line bundle L on a smooth projective variety is ample if and only if $\varepsilon(X, L, x) > 0$ for every $x \in X$.

Now define

$$\varepsilon(X, L, r) := \max_{x_1, \ldots, x_r \in X} \varepsilon(X, L, x_1, \ldots, x_r).$$

It is known that $\varepsilon(X, L, r)$ is attained at a *very general* set of points $x_1, \ldots, x_r \in X$; see [20]. This means that $\varepsilon(X, L, r) = \varepsilon(X, L, x_1, \ldots, x_r)$ for all $(x_1, \ldots, x_r)$ outside some countable union of proper Zariski closed sets in $X^r = X \times X \times \ldots \times X$.

Date: May 31, 2017.

2010 *Mathematics Subject Classification.* Primary 14C20; Secondary 14H50.

Authors were partially supported by a grant from Infosys Foundation.

The following is a well-known upper bound for Seshadri constants. Let n be the dimension of X. Then for any $x_1, \ldots, x_r \in X$,

$$\varepsilon(X, L, x_1, \ldots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}.$$

In view of this upper bound, a lot of research is aimed at finding good lower bounds for the Seshadri constants of ample line bundles, primarily when X is a surface. There has been extensive work on computing or finding lower bounds for Seshadri constants on surfaces, mainly in the single point case ($r = 1$). The multi-point case ($r \geq 2$) is also of interest and there are results in this case in various situations. Some results for multi-point Seshadri constants can be found in [3, 9, 10, 13, 14, 15, 22, 24, 25].

Let X be a surface and let L be an ample line bundle on X. One of the crucial ideas in finding lower bounds is the observation that if a Seshadri constant $\varepsilon(X, L, x_1, \ldots, x_r)$ is *sub-maximal* (i.e., $\varepsilon(X, L, x_1, \ldots, x_r) < \sqrt{L^2/r}$), then there is actually an irreducible and reduced curve C which passes through at least one of the points x_i such that $\varepsilon(X, L, x_1, \ldots, x_r) = \frac{L \cdot C}{\sum_i \mathrm{mult}_{x_i}(C)}$. Such curves are called *Seshadri curves*. See [5, Proposition 1.1] for a proof of their existence for sub-maximal single-point Seshadri constants which generalizes easily to the multi-point case.

In our first main result, Theorem 2.1, we consider an arbitrary smooth complex projective surface X and an ample line bundle L on X, and show that, for an integer $r \geq 2$, the Seshadri constant satisfies $\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, or there is a Seshadri curve C on X passing through $s \leq r$ very general points with multiplicity one at each point. This bound is a generalization of [13, Theorem 2.1], where it was proved for surfaces with Picard number 1.

In some situations, it is possible to either rule out the existence of a Seshadri curve passing through very general points with multiplicity one each, or limit the possibilities for such curves. We illustrate this phenomenon in a few cases. See Example 2.5 and Lemma 3.6.

In Section 3, we study multi-point Seshadri constants on ruled surfaces. Motivated by the results of [11, 21], where single-point Seshadri constants are studied on ruled surfaces, we make precise computations of multi-point Seshadri constants (Theorem 3.1) when the number of points r is small. Then we study the case of rational ruled surfaces in greater detail. We show in Theorem 3.8 that, given a rational ruled surface X and an ample line bundle L on X, there exists a large enough r (depending on L) such that there are no Seshadri curves for L passing through r very general points with multiplicity one each.

Let (X, L) be a polarized surface (that is, X is a surface and L is an ample line bundle on X). The *Nagata-Biran-Szemberg Conjecture* predicts that for large enough r, the Seshadri constant $\varepsilon(X, L, r)$ is maximal; i.e, $\varepsilon(X, L, r) = \sqrt{\frac{L^2}{r}}$. In fact, the conjecture makes a precise prediction about how large r should be. It says that the maximality holds for $r \geq k_0^2 L^2$, where k_0 is the smallest integer such that the linear system $|k_0 L|$ contains smooth non-rational curves. Note that $k_0 = 3$ when $(X, L) = (\mathbb{P}^2, \mathcal{O}(1))$. In this sense, the Nagata-Biran-Szemberg Conjecture is a generalization of the celebrated *Nagata Conjecture* (see [19, Remark 5.1.24], [22], or [23] for more details). The Nagata-Biran-Szemberg Conjecture has been verified asymptotically in [22, Corollary 4.4].

The results in this paper provide further support for the Nagata-Biran-Szemberg conjecture. For small r, specifically for $r < k_0^2 L^2$, the Seshadri constant may be small, but not for $r \geq k_0^2 L^2$. This is evident from our results on ruled surfaces. In Theorem 3.1, we show that the Seshadri constants are small on a ruled surface when r is small. On a rational ruled surface, Theorem 3.8 shows that $\varepsilon(X, L, r)$ asymptotically approaches the maximal value as r increases.

All the varieties we consider are defined over $\mathbb{C}$, the field of complex numbers. A *surface* is a two-dimensional smooth complex projective variety.

Acknowledgements: We sincerely thank Brian Harbourne, D. S. Nagaraj and Tomasz Szemberg for useful discussions. We also thank the referee for pointing out a way to strengthen Theorem 2.1 and numerous other suggestions which improved the exposition.

2. A general lower bound

In this section, we prove a general lower bound for multi-point Seshadri constants on any surface. This is a generalization of [13, Theorem 2.1] where the same bound is proved for surfaces with Picard number 1.

Theorem 2.1. *Let X be a surface and let L be an ample line bundle on X. Let $r \geq 2$ be an integer. If $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, then $\varepsilon(X, L, r) = \frac{L\cdot C}{s}$, where C is an irreducible and reduced curve on X which passes through $s \leq r$ very general points with multiplicity one at each point.*

Moreover, for every $r \geq 2$, there exists a polarized surface (X, L) such that $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$.

Proof. Suppose that $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$. In particular, the Seshadri constant $\varepsilon(X, L, r)$ is not maximal. Then, as we noted in the introduction, there exists an irreducible and reduced curve C such that $\varepsilon(X, L, r) = \frac{L\cdot C}{\sum_i m_i}$, where $m_1, \ldots, m_r$ are the multiplicities of C at some very general points $x_1, \ldots, x_r$.

Arrange the multiplicities in decreasing order, so that $m_1 \geq m_2 \geq \ldots \geq m_s > 0$ and $m_{s+1} = \ldots = m_r = 0$ for some $1 \leq s \leq r$.

Since the points $x_1, \ldots, x_r$ are very general, there is a non-trivial one-parameter family of irreducible and reduced curves $\{C_t\}_{t\in T}$ parametrized by some smooth curve T and containing points $x_{1,t}, \ldots, x_{r,t} \in C_t$ with $\text{mult}_{x_{i,t}}(C_t) \geq m_i$ for all $1 \leq i \leq r$ and $t \in T$.

By a result of Ein-Lazarsfeld [8] and Xu [26, Lemma 1], we have

$$C^2 \geq m_1^2 + \ldots + m_s^2 - m_s. \quad (2.1)$$

First, suppose that $m_1 = 1$. Hence $m_1 = \ldots = m_s = 1$. In this case, the Seshadri constant $\varepsilon(X, L, r)$ is computed by a curve C which passes through $s \leq r$ very general points with multiplicity one each.

Now we assume that $m_1 \geq 2$ and show that $\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, contradicting our assumption.

We will apply the following special case of [12, Lemma 2.3], which holds when either $m_1 \geq 2$ and $s \geq 3$ or if $s = 2$ then $(m_1, m_2) \neq (2, 2)$.

$$\frac{(s+3)s}{s+2}\left(\sum_{i=1}^{s} m_i^2 - m_s\right) \geq \left(\sum_{i=1}^{s} m_i\right)^2. \tag{2.2}$$

First, suppose that the conditions required for (2.2) hold. That is: $m_1 \geq 2$, $s \geq 3$ or if $s = 2$ then $(m_1, m_2) \neq (2, 2)$.

We have $(L \cdot C)^2 \geq L^2C^2$, by the Hodge Index Theorem. Hence the inequalities (2.1) and (2.2) give

$$L \cdot C \geq \sqrt{L^2}\left(\sum_{i=1}^{s} m_i\right)\sqrt{\frac{s+2}{s(s+3)}} \geq \left(\sum_{i=1}^{s} m_i\right)\sqrt{\frac{L^2}{r}}\sqrt{\frac{r+2}{r+3}}.$$

Thus

$$\varepsilon(X, L, r) = \frac{L \cdot C}{\sum_{i=1}^{s} m_i} \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

Next, suppose that $s = 2$ and $(m_1, m_2) = (2, 2)$. In this case, we use an inequality that is stronger than (2.1). With the notation as in (2.1), if $m_1 \geq 2$, then we have

$$C^2 \geq m_1^2 + \ldots + m_s^2 - m_1 + \mathrm{gon}(\tilde{C}). \tag{2.3}$$

Here $\tilde{C}$ is the normalization of C and $\mathrm{gon}(\tilde{C})$ is the *gonality* of $\tilde{C}$ which is defined to be the least degree of a covering $\tilde{C} \to \mathbb{P}^1$. See [2, Lemma 2.1] and [18, Theorem A] for the single-point case and [9, Lemma 2.12] for the multi-point case.

Since $\mathrm{gon}(\tilde{C})$ is a positive integer, (2.3) gives $C^2 \geq 7$ in our situation. Now, using the Hodge Index Theorem as above, we get

$$\varepsilon(X, L, r) = \frac{L \cdot C}{4} \geq \sqrt{L^2}\sqrt{7/16} \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

The last inequality holds because $\frac{r+2}{r(r+3)} \leq 7/16$ for $r \geq 2$.

Finally, let $s = 1$ and $m = m_1$. By (2.1), $C^2 \geq m^2 - m$. Since $r \geq 2$, by hypothesis, we have $\frac{r+2}{r(r+3)} \leq 2/5$. Further, $m^2 - m \geq 2m^2/5$ for $m \geq 2$. So by the Hodge Index Theorem as above, we have

$$\varepsilon(X, L, r) = \frac{L \cdot C}{m} \geq \sqrt{L^2}\sqrt{2/5} \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}.$$

For the last statement of the theorem, see Example 2.2. □

Example 2.2. In this example, we show that for every $r \geq 2$, there exists a polarized surface (X, L) such that $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$.

For $r = 2$, take $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Then $\varepsilon(X, L, 2) = 1/2 < \sqrt{4/5 \cdot 1/2}$.

Let $r \geq 3$. Choose integers $n > e \geq 0$ such that $r = 2n - e + 1$. Let X be the ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with a normalized section C_0 and a fibre f. Let $\pi : X \to \mathbb{P}^1$ be the canonical map. Let $L = C_0 + nf$. Then L is very ample by [17, Chapter V, Theorem 2.17].

We have $L^2 = 2n - e = r - 1$. Note that $K_X = -2C_0 + (-2 - e)f$ and $H^1(X, L) = H^1(\mathbb{P}^1, \pi_\star(L)) = 0$. Now it is easy to calculate, using Riemann-Roch, that $h^0(X, L) = r + 1$. Hence given any r points $x_1, \ldots, x_r \in X$, there is an effective divisor D passing through $x_1, \ldots, x_r$ such that D is linearly equivalent to L.

So $\varepsilon(X, L, r) \leq \frac{r-1}{r} < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$. In fact, we claim that $\varepsilon(X, L, r) = \frac{r-1}{r}$. First note that, by Theorem 2.1, $\varepsilon(X, L, r)$ is computed by a curve C passing through $s \leq r$ very general points with multiplicity one each. Then $C^2 \geq s - 1$, by (2.1). If $s = 1$ then $\varepsilon(X, L, r) = L \cdot C \geq 1$, which is impossible since $\varepsilon(X, L, r) \leq \frac{r-1}{r} < 1$. So let $s \geq 2$. By the Hodge Index Theorem, $L \cdot C \geq \sqrt{L^2C^2} \geq \sqrt{(r-1)(s-1)}$. Thus

$$\varepsilon(X, L, r) = \frac{L \cdot C}{s} \geq \sqrt{\frac{(r-1)(s-1)}{s^2}} \geq \frac{r-1}{r}.$$

Note that the embedding of X in $\mathbb{P}^r$ determined by L is a rational normal scroll. This example is also discussed in [22, Example 4.2]. In subsection 3.2, we study Seshadri constants on rational ruled surfaces in more detail.

Remark 2.3. We may compare our Theorem 2.1 with [22, Theorem 4.1]. This result says that if $\varepsilon(X, L, r) < \sqrt{\frac{r-1}{r}}\sqrt{\frac{L^2}{r}}$, then X has a fibration by Seshadri curves. In situations where the existence of a Seshadri curve C passing through $s \leq r$ points with multiplicity one each can be ruled out (see Remark 2.4 for one such instance), the bound $\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$ holds, by Theorem 2.1. This is a better bound than the one given in [22, Theorem 4.1].

Remark 2.4. Suppose that for a polarized surface (X, L) and an integer $r \geq 2$, we have $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$. Then by Theorem 2.1, $\varepsilon(X, L, r) = \frac{L \cdot C}{s}$ for a curve C passing through $s \leq r$ very general points with multiplicity one each. Then C must satisfy $C^2 < s$, as we show now. Suppose that $C^2 \geq s > 0$. By the Hodge Index Theorem, $L \cdot C \geq \sqrt{L^2C^2}$. So $\varepsilon(X, L, r) = \frac{L \cdot C}{s} \geq \sqrt{\frac{L^2}{r}}\sqrt{\frac{C^2}{s}} \geq \sqrt{\frac{L^2}{r}}$. So $\varepsilon(X, L, r)$ attains the maximal possible value and this contradicts our assumption.

In view of this remark, it is useful to investigate the following property: If an irreducible and reduced curve C on X passes through r very general points, then $C^2 \geq r$.

If we have some information about curves C with the above property on a surface X, it may be possible to establish that lower the bound in Theorem 2.1 holds. For instance, see Example 2.5, Lemma 3.6 and Theorem 3.8.

Example 2.5. Let X be a K3 surface. Then $h^1(\mathcal{O}_X) = 0$, $h^2(\mathcal{O}_X) = 1$ and $K_X = \mathcal{O}_X$. Let C be an irreducible curve on X. Then $h^2(\mathcal{O}_X(C)) = 0$. Further, $h^1(\mathcal{O}_X(C)) = h^1(\mathcal{O}_X(-C))$, by Serre duality. Taking cohomology of the exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we see that $h^1(\mathcal{O}_X(-C)) = 0$. For more details on K3 surfaces, see [1, Chapter VIII] or [6, Chapter VIII].

So if C is an irreducible curve on X, then $h^0(\mathcal{O}_X(C)) = \frac{C^2}{2} + 2$, by Riemann-Roch. It is easy to see that C passes through r very general points if and only if $h^0(\mathcal{O}_X(C)) \geq r + 1$. Thus if C passes through r very general points, then $C^2 \geq 2r - 2$. So if $r \geq 2$, it follows that $C^2 \geq r$.

Now let L be an ample line bundle on X and let $r \geq L^2$. Suppose that there exists a Seshadri curve for L passing through $s \leq r$ very general points with multiplicity one each. If $s \geq 2$, then by the argument in the previous paragraph $C^2 \geq s$. So Remark 2.4 shows that $\varepsilon(X, L, r) = \sqrt{\frac{L^2}{r}}$. If $s = 1$, then $\varepsilon(X, L, r) = L \cdot C \geq 1$. On the other hand, $\sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}} < 1$, if $r \geq L^2$.

Hence for an ample line bundle L on a K3 surface X, we have $\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, for $r \geq \max\{L^2, 2\}$, by Theorem 2.1 and Remark 2.4.

3. Ruled surfaces

Let C be a smooth curve and let $\pi : X \to C$ be a ruled surface over C. We choose a normalized vector bundle E of rank 2 on C such that $X \cong \mathbb{P}(E)$. Let $e = -\deg(E)$. Let C_0 be the image of a section of π such that $C_0^2 = -e$ and let f be a fibre of π. Then the Picard group of X modulo numerical equivalence is a free abelian group of rank 2 generated by C_o and f. We have $f^2 = 0$ and $C_0 \cdot f = 1$.

A complete characterization of ample line bundles on X is known. For this, and other details on ruled surfaces, we refer to [17, Chapter V, Section 2].

In this section, we consider a ruled surface $\pi : X \to C$ and an ample line bundle L on X. For an integer $r \geq 1$ and points $x_1, \ldots, x_r \in X$, we are interested in the problem of computing the Seshadri constants $\varepsilon(X, L, x_1, \ldots, x_r)$, or obtaining lower bounds for the general Seshadri constant $\varepsilon(X, L, r)$.

3.1. The case $r \leq e$.

Let $X \to C$ be a ruled surface with invariant $e \in \mathbb{Z}$. In this subsection, we will assume that $1 \leq r \leq e$. We are primarily motivated by [11, Theorems 4.1, 4.2] and [21, Theorem 3.27] which compute the single-point Seshadri constants on ruled surfaces. Our main result in this case is Theorem 3.1, which generalizes these results to the multi-point case when $e > 0$.

Theorem 3.1. *Let X be a ruled surface with invariant $e > 0$. Fix a positive integer $r \leq e$. Let L be an ample line bundle on X and let $x_1, \ldots, x_r \in X$. Denote by t the maximum number of points among $x_1, \ldots, x_r$ that lie on a single fibre f and by s the number of points among $x_1, \ldots, x_r$ lying on C_0.*

Then the Seshadri constant $\varepsilon(X, L, x_1, \dots, x_r) = \min\left\{\frac{L\cdot f}{t}, \frac{L\cdot C_o}{s}\right\}$*, if* $s > 0$*. Otherwise,* $\varepsilon(X, L, x_1, \dots, x_r) = \frac{L\cdot f}{t}$*.*

Proof. Let $L = aC_0 + bf$. By [17, Chapter V, Proposition 2.20], $a > 0$ and $b > ae$. Let $D = \alpha C_0 + \beta f$ be any irreducible and reduced curve on X such that $D \neq C_0$ and $D \neq f$. In this case, we have $\alpha > 0$ and $\beta \geq \alpha e$, by the same reference as above.

Suppose that D passes through $x_1, \dots, x_r$ with multiplicities $m_1, \dots, m_r$. Assume that $m := \sum_{i=1}^r m_i > 0$. We will show that $\frac{L\cdot D}{m} \geq \frac{L\cdot f}{t}$ and the theorem will follow.

Since $D \neq f$, $\alpha = D \cdot f \geq m_i$ for all i. This follows by considering a fibre through the point x_i. Hence $r\alpha \geq m$. Note that $b \geq ae + 1$ and $\beta \geq \alpha e$. So

$$\frac{L\cdot D}{m} = \frac{-a\alpha e + \alpha b + a\beta}{m} \geq \frac{\alpha + a\alpha e}{m} \geq \frac{\alpha(1+ae)}{m} \geq \frac{ae+1}{r}.$$

Since $e \geq r$, $\frac{L\cdot D}{m} \geq \frac{ae+1}{r} \geq \frac{a}{t} = \frac{L\cdot f}{t}$. □

Corollary 3.2. *With* $X, e, x_1, \dots, x_r, t$ *and* s *as in Theorem 3.1, let* $L = aC_0 + bf$ *be an ample line bundle on* X *such that* $b \geq 2ae + 1$*. Then* $\varepsilon(X, L, x_1, \dots, x_r) = \frac{L\cdot f}{t}$*.*

Proof. Assume that $s > 0$, as otherwise there is nothing to prove. We will prove that $\frac{L\cdot C_0}{s} \geq \frac{L\cdot f}{t}$. We saw above in the proof of Theorem 3.1 that $\frac{ae+1}{r} \geq \frac{L\cdot f}{t}$. So it suffices to show that $\frac{ae+1}{r} \leq \frac{L\cdot C_0}{s}$.

If not, we have $\frac{ae+1}{r} > \frac{L\cdot C_0}{s}$, which implies that $s > \left(\frac{b-ae}{ae+1}\right) r \geq r$, since $b \geq 2ae + 1$. But this is absurd, since there are only r points. □

Given a positive rational number q, we show next that there exists a surface X and an ample line bundle L on X such that for r sufficiently large, there are points $x_1, \dots, x_r \in X$ with $\varepsilon(X, L, x_1, \dots, x_r) = q$. In particular, this shows that multi-point Seshadri constants for arbitrarily large r can be arbitrarily small. Compare this with Miranda's example [19, Example 5.2.1] which shows that single-point Seshadri constants can be arbitrarily small.

Corollary 3.3. *Let* $q = \frac{a}{t}$ *be a rational number with* $a, t > 0$*. Then there exist a polarized ruled surface* (X, L)*, an integer* r *and points* $x_1, \dots, x_r \in X$ *such that* $\varepsilon(X, L, x_1, \dots, x_r) = q$*.*

Proof. Let $r = e = t$. Consider a ruled surface X with invariant e. For specificity, we may consider the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$. Let $L = aC_0 + (2ae + 1)f$. Then L is ample. Choose points $x_1, \dots, x_r$ on X so that the maximum number of points lying on a single fibre is t. Then, by Corollary 3.2, $\varepsilon(X, L, x_1, \dots, x_r) = a/t$. □

Note that in the above corollary, $L^2 = 3a^2e + 2a$, so that $\sqrt{\frac{L^2}{r}} \geq a \geq 1$. Thus the smallness of the Seshadri constant $\varepsilon(X, L, x_1, \dots, x_r)$ is *not* due to the smallness of $\sqrt{\frac{L^2}{r}}$. Contrast this with the case of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

While the multi-point Seshadri constant $\varepsilon(X, L, x_1, \ldots, x_r)$ can be arbitrarily small at special points $x_1, \ldots, x_r$, its value at very general points is always $L \cdot f$, as we show now.

Corollary 3.4. *Let X be a ruled surface with invariant e and let $1 \leq r \leq e$. For an ample line bundle L, one has $\varepsilon(X, L, r) = L \cdot f$.*

Proof. Note that $\varepsilon(X, L, r)$ is attained at r very general points $x_1, \ldots, x_r \in X$. Since $C_0^2 = -e < 0$, no very general point lies on C_0. Let f be the fibre through x_1. Since the points $x_1, \ldots, x_r$ are very general, $x_i \notin f$ for $i = 2, \ldots, r$. Thus in the notation of Theorem 3.1, $s = 0$ and $t = 1$. Hence $\varepsilon(X, L, r) = \varepsilon(X, L, x_1, \ldots, x_r) = L \cdot f$. □

Remark 3.5. Let X be a ruled surface with invariant e and let $1 \leq r \leq e$. Then it follows from Corollary 3.4 that the Seshadri constant $\varepsilon(X, L, r)$ is sub-maximal for *any* ample line bundle L on X. Indeed, let $L = aC_0 + bf$. Since L is ample, $b > ae$. So $L^2 = -a^2e + 2ab = a(2b - ae) > a^2e$. So $\frac{L^2}{r} > \frac{a^2e}{r} \geq a^2$. Hence $a = \varepsilon(X, L, r) < \sqrt{L^2/r}$.

For any r, if $e \gg r$, then $\varepsilon(X, L, r) = a$ would be very small compared to $\sqrt{\frac{L^2}{r}} = \sqrt{\frac{2ab - a^2e}{r}}$; in particular, smaller than $\sqrt{\frac{r-1}{r}}\sqrt{\frac{L^2}{r}}$. By [22, Theorem 4.1], it follows that X is fibred by Seshadri curves. Of course, in this particular case this is obvious, since the fibres f are Seshadri curves.

3.2. **Rational ruled surfaces.** In this subsection, we will consider rational ruled surfaces and clarify the bound given in Theorem 2.1.

Let X be a ruled surface over $\mathbb{P}^1$. We fix a normalized vector bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$ such that $X = \mathbb{P}(E)$ and $e \geq 0$. Denote by C_0 a section with $C_0^2 = -e$ and let f be a fibre.

First, we make the following observation.

Lemma 3.6. *Let X be a rational ruled surface. Let C be an irreducible and reduced curve on X passing through $r \geq 1$ very general points. If $C^2 < r$, then $C^2 = r - 1$ and C is a smooth rational curve.*

Proof. By (2.1), we have $C^2 \geq r - 1$, in any case. So $C^2 = r - 1$. We also note that C passes through r very general points if any only if $h^0(X, \mathcal{O}_X(C)) \geq r + 1$. This follows from a simple dimension count on the linear system $|C|$ and the fact the very general points impose independent conditions. In our situation, note also that $h^0(X, \mathcal{O}_X(C)) = r + 1$, because if $h^0(X, \mathcal{O}_X(C)) > r + 1$, then C passes through more than r very general points and $C^2 \geq r$ by (2.1).

Since C is effective, $h^2(X, \mathcal{O}_X(C)) = 0$. Using the projection formula to push-down $\mathcal{O}_X(C)$ to $\mathbb{P}^1$, we see that $h^1(X, \mathcal{O}_X(C)) = 0$. Now since $h^0(X, \mathcal{O}_X(C)) = r + 1$, the Riemann-Roch theorem gives $K_X \cdot C = -r - 1$ and by adjunction, $2p_a(C) - 2 = C^2 + K_X \cdot C = -2$. Hence $p_a(C) = 0$ and C is a smooth rational curve. □

Lemma 3.7. *Let X be a rational ruled surface with invariant $e \geq 0$, normalized section C_0 and a fibre f. Let $C = mC_0 + nf$ be an irreducible smooth curve on X. Then C is rational if and only if*

(1) $C = C_0$, *or*
(2) $C = f$, *or*
(3) $m = 1, n > e$, *or*
(4) $e > 0, m = 1, n = e$, *or*
(5) $e = 0, m \geq 1, n = 1$, *or*
(6) $e = 1, m = n = 2$.

Proof. In each of the cases listed above, it is easy to see, using the adjunction formula, that C is rational.

For the converse, let C be an irreducible, smooth rational curve and suppose that $C \neq C_0, C \neq f$. By [17, Chapter V, Corollary 2.8], $m > 0, n > me$ or $e > 0, m > 0, n = me$.

We have $C^2 = -m^2e + 2mn$ and $K_X \cdot C = me - 2n - 2m$. By adjunction,

$$-2 = C^2 + K_X \cdot C = -m^2e + me - 2m + n(2m - 2). \tag{3.1}$$

Case 1: $n > me$.

So $-2 = -m^2e + me - 2m + n(2m - 2) \geq -m^2e + me - 2m + (me + 1)(2m - 2) = m^2e - me - 2 = me(m - 1) - 2$. So $0 \geq me(m - 1)$. Since $m \geq 1, e \geq 0$, we have $m = 1$ or $e = 0$. If $m = 1$, then we have case (3).

Let $e = 0$. Then $-2 = 2(mn - m - n)$, by (3.1). So either $m = 1$ or $n = 1$. If $m = 1$, we have case (3). If $n = 1$, we have case (5).

Case 2: $n = me$.

In this case, $e > 0$. By (3.1), $-2 = (m^2 - m)e - 2m$. Hence $2m - 2 = (m^2 - m)e \geq m^2 - m$. Hence $m = 1$ or $m = 2, e = 1$. If $m = 1$, we have case (4) and if $m = 2, e = 1$, we have case (6). □

Now we state our main theorem on rational ruled surfaces.

Theorem 3.8. *Let X be a rational ruled surface. Let L be an ample line bundle on X. Then*

$$\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}, \text{ for } r \geq L^2 + 5.$$

Proof. If the statement of the theorem is false, then by Theorem 2.1, there is an irreducible and reduced curve C passing through $s \leq r$ very general points with multiplicity one each such that $\varepsilon(X, L, r) = \frac{L \cdot C}{s}$. We will show this is impossible.

By Remark 2.4, C must satisfy $C^2 = s - 1$. By Lemma 3.6, C is a smooth rational curve. We will consider four cases below which deal with all the six possibilities listed in Lemma 3.7.

Let $L = aC_0 + bf$ with $a > 0$ and $b > ae$.

Case 1: Let $C = C_0$. Then $C^2 = -e \leq 0$. On the other hand, $C^2 = s - 1 \geq 0$. So $e = 0$ and $s = 1$. Then $\varepsilon(X, L, r) = L \cdot C_0 \geq 1$. But this is a contradiction because we have $r \geq L^2$ and $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}} < 1$. The same argument holds when $C = f$.

Case 2: Let $C = C_0 + nf$ for $n \geq e$.

Then $L \cdot C = an + b - ae$ and $C^2 = 2n - e = s - 1$. So $\varepsilon(X, L, r) = \frac{L \cdot C}{s} = \frac{an+b-ae}{2n-e+1} \geq \frac{an+1}{2n-e+1}$. The last inequality holds because $b > ae$. Thus if $a \geq 2$, $\varepsilon(X, L, r) = \frac{L \cdot C}{s} \geq 1$, again contradicting our assumption.

Now let $a = 1$. The desired contradiction follows from the inequality:

$$\frac{(L \cdot C)^2}{s^2} \geq \left(\frac{r+2}{r+3}\right)\frac{L^2}{r}. \tag{3.2}$$

We will now establish (3.2). If $s = 1$, then $\varepsilon(X, L, r) \geq 1$. So there is nothing to prove.

Let $2 \leq s < r$. By the Hodge Index Theorem, (3.2) follows if we show that $\frac{C^2}{s^2} \geq \frac{r+2}{r(r+3)}$. This, in turn, is equivalent to $r(r+3)(s-1) \geq (r+2)s^2$. It is not difficult to check this inequality holds when $2 \leq s \leq r - 1$ and $r \geq 4$. Indeed, writing the difference of the two terms as a quadratic in s, we have $Q(s) = -(r+2)s^2 + r(r+3)s - r(r+3)$. Its graph is a downward sloping parabola and it is easy to see that $Q(2) \geq 0$ and $Q(r-1) \geq 0$ when $r \geq 4$.

Now let $s = r$. By hypothesis, $4 \leq r - 1 - L^2 = C^2 - L^2 = 2(n-b)$. Hence $n - b \geq 2$. Note that $L^2 = 2b - e$, $C^2 = 2n - e = r - 1$ and $L \cdot C = b + n - e$. Thus $(L \cdot C)^2 - L^2C^2 = (n-b)^2$. We also have $C^2 = L^2 + 2(n-b)$ and $L^2 = C^2 - 2(n-b) = r - 1 - 2(n-b)$.

$$\begin{aligned}
(3.2) \quad &\Leftrightarrow \quad \frac{(L \cdot C)^2}{r^2} \geq \left(\frac{r+2}{r+3}\right)\frac{L^2}{r} \\
&\Leftrightarrow \quad L^2C^2 + (n-b)^2 \geq \left(\frac{r(r+2)}{r+3}\right)L^2 \\
&\Leftrightarrow \quad (L^2)^2 + 2(n-b)L^2 + (n-b)^2 \geq \left(\frac{r(r+2)}{r+3}\right)L^2 \\
&\Leftrightarrow \quad L^2 + 2(n-b) + \frac{(n-b)^2}{L^2} \geq \frac{r(r+2)}{r+3} \\
&\Leftrightarrow \quad r - 1 + \frac{(n-b)^2}{L^2} \geq \frac{r(r+2)}{r+3} \\
&\Leftrightarrow \quad \frac{(n-b)^2(r+3)}{L^2} \geq 3.
\end{aligned}$$

The last inequality holds because $n - b \geq 2$ and $r \geq L^2 + 5$.

Case 3: Let $e = 0, m \geq 1$ and $n = 1$.

Then $C^2 = 2m$ and $s = 2m+1$. So $\frac{L\cdot C}{s} = \frac{a+bm}{2m+1}$. If $b \geq 2$, then $\varepsilon(X,L,r) = \frac{L\cdot C}{s} \geq 1$, which is a contradiction. Hence we have $b = 1$ and $L = aC_0 + f$. The argument here is very similar to the argument in **Case 2** with $e = 0$.

Case 4: Finally, let $C = 2C_0 + 2f$ and $e = 1$. We will show that C can not be a Seshadri curve in this case.

We have $C^2 = 4$, $s = 5$. Now $L \cdot C = 2b$ and $\frac{L\cdot C}{s} = \frac{2b}{5}$. If $a \geq 2$, then $b \geq 3$. So $\varepsilon(X,L,r) = \frac{L\cdot C}{s} \geq 1$, which is a contradiction because $r \geq L^2$. So $a = 1$ and $b = 2$. Thus $\frac{L\cdot C}{s} = 4/5$. On the other hand, $\sqrt{\frac{L^2}{r}}\sqrt{\frac{r+2}{r+3}} = \sqrt{\frac{3(r+2)}{r(r+3)}}$. It is easy to see that $4/5 \geq \sqrt{\frac{3(r+2)}{r(r+3)}}$ for all $r \geq 5$.

This completes the proof of the theorem. □

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