SESHADRI CONSTANTS ON SURFACES WITH PICARD NUMBER 1

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ABSTRACT. Let X be a smooth projective surface with Picard number 1. Let L be the ample generator of the Néron-Severi group of X. Given an integer $r \geq 2$, we prove lower bounds for the Seshadri constant of L at r very general points in X.

1. Introduction

Seshadri constants are a local measure of positivity of a line bundle on a projective variety. They arose out of an ampleness criterion of Seshadri, [12, Theorem 7.1] and were defined by Demailly, [5]. They have become an important area of research with interesting connections to other areas of mathematics. For a comprehensive account of Seshadri constants, see [2].

Let X be a smooth projective variety and let L be a nef line bundle on X. Let $r \geq 1$ be an integer. For $x_1, \ldots, x_r \in X$, the *Seshadri constant* of L at $x_1, \ldots, x_r$ is defined as follows.

$$\varepsilon(X, L, x_1, \ldots, x_r) := \inf_{C \cap \{x_1, \ldots, x_r\} \neq \emptyset} \frac{L \cdot C}{\sum\limits_{i=1}^{r} \mathrm{mult}_{x_i} C}.$$

We remark that the infimum above is the same as the infimum taken over irreducible, reduced curves C such that $C \cap \{x_1, \ldots, x_r\} \neq \emptyset$. Indeed, we have the inequality

$$\frac{L \cdot (C+D)}{\sum_{i=1}^{r} \mathrm{mult}_{x_i}(C) + \sum_{i=1}^{r} \mathrm{mult}_{x_i}(D)} \geq \min\left(\frac{L \cdot C}{\sum_{i=1}^{r} \mathrm{mult}_{x_i}(C)}, \frac{L \cdot D}{\sum_{i=1}^{r} \mathrm{mult}_{x_i}(D)}\right).$$

The Seshadri criterion for ampleness says that a line bundle L on a projective variety is ample if and only if $\varepsilon(X, L, x) > 0$ for every $x \in X$.

The following is a well-known upper bound for Seshadri constants. Let n be the dimension of X. Then for any $x_1, \ldots, x_r \in X$,

$$\varepsilon(X, L, x_1, \ldots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}.$$

Next one defines

$$\varepsilon(X, L, r) := \max_{x_1, \ldots, x_r \in X} \varepsilon(X, L, x_1, \ldots, x_r).$$

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It is known that $\varepsilon(X, L, r)$ is attained at a very general set of points $x_1, \ldots, x_r \in X$; see [15]. Here *very general* means that $(x_1, \ldots, x_r)$ is outside a countable union of proper Zariski closed sets in $X^r = X \times X \times \ldots \times X$.

In this paper we will be concerned with lower bounds for $\varepsilon(X, L, r)$ when X is a smooth projective surface with Picard number 1, L is the ample generator of the Néron-Severi group of X (see definitions below) and $r \geq 2$ is an integer.

There has been extensive work on lower bounds for $\varepsilon(X, L, 1)$.

Let X be an arbitrary smooth complex projective surface and let L be an ample line bundle on X. It is easy to see that if L is very ample then $\varepsilon(L, x) \geq 1$ for any point $x \in X$. On the other hand, if L is ample, but not very ample, it is possible that $\varepsilon(L, x) < 1$. In fact, Seshadri constants can be arbitrarily small. Miranda has shown that given any *rational* number $\varepsilon > 0$, there exist a surface X, an ample line bundle L and a point $x \in X$ such that $\varepsilon(X, L, x) = \varepsilon$; see [14, Example 5.2.1]. In Miranda's construction, the surface X typically has a large Picard number. It is possible that $\varepsilon(X, L, x) < 1$ even on a surface with Picard number 1; see [3, Example 1.2].

However, in an important paper [6], Ein and Lazarfeld proved that $\varepsilon(X, L, 1) \geq 1$, if L is ample. In fact, they prove the following theorem.

Theorem 1.1 (Ein-Lazarsfeld). *Let X be a smooth projective surface and let L be an ample line bundle on X. Then $\varepsilon(X, L, x) \geq 1$ for all except possibly countably many points $x \in X$. Further, if $L^2 > 1$, the set of exceptional points is actually finite.*

There are many other results calculating $\varepsilon(X, L, 1)$ or giving bounds for it. We will just mention a few that are of relevance to us here. Bauer [1] considered several cases of surfaces including abelian surfaces of Picard number 1. Szemberg, in [20] and [21], dealt with surfaces of Picard number 1. Szemberg has a conjecture for a lower bound for $\varepsilon(X, L, 1)$ for surfaces with Picard number 1; see [20, Conjecture]. A recent preprint [7] gives new lower bounds for $\varepsilon(X, L, 1)$ for an arbitrary surface and an ample line bundle L. In the process, it verifies the conjecture in [20] in many cases.

Contrary to the case of $\varepsilon(X, L, 1)$, there are not too many lower bounds for $\varepsilon(X, L, r)$ in the literature when $r \geq 2$. We mention some results in this case.

Küchle [13] considers multi-point Seshadri constants on an arbitrary projective variety. Syzdek and Szemberg [18] prove a lower bound for multi-point Seshadri constants on any surface.

For an arbitrary surface X and a nef and big line bundle L on X, [10, Theorem 1.2.1] gives good lower bounds for $\varepsilon(X, L, r)$ in terms of the degree of the least degree curve passing through r general points with a given multiplicity m. Using this theorem, [10, Corollary 1.2.2] has a bound for $\varepsilon(X, L, r)$ when L is the ample generator the Néron-Severi group of X.

For an arbitrary surface X, when L is very ample, [9, Theorem I.1] gives lower bounds for $\varepsilon(X, L, r)$.

When X has Picard number 1 and L is the ample generator of the Néron-Severi group, [20, Theorem 3.2] also gives lower bounds for $\varepsilon(X, L, r)$. [1, Section 8] considers multi-point Seshadri constants for abelian surfaces.

Another method of obtaining lower bounds for $\varepsilon(X, L, r)$ comes from the following bound (see [4, 16, 17]):

$$\varepsilon(X, L, r) \geq \varepsilon(X, L, 1)\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r). \tag{1.1}$$

The well-known Nagata Conjecture is a statement about $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r)$. It says that, for $r \geq 10$, $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r) = \frac{1}{\sqrt{r}}$. This was proved by Nagata when r is a square and is open in all other cases. But there are several results which give good bounds close to the bound expected by the Nagata Conjecture (see [9, 11]). Using known bounds for $\varepsilon(X, L, 1)$, some of which were mentioned above, and bounds on $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r)$, one can get bounds on $\varepsilon(X, L, r)$ by (1.1).

Let X be a smooth projective surface over $\mathbb{C}$. The *Picard group* of X, denoted Pic(X), is the group of isomorphism classes of line bundles on X. The *divisor class group* of X, denoted Div(X), is the group of divisors on X modulo linear equivalence. Then Pic(X) and Div(X) are isomorphic as abelian groups. The *Néron-Severi* group of X is defined as

$$\mathrm{NS(X)} := \mathrm{Pic(X)}/\mathrm{Pic}^0\mathrm{(X)},$$

where $\mathrm{Pic}^0\mathrm{(X)}$ is the subgroup of Pic(X) consisting of line bundles which are algebraically equivalent to the trivial line bundle. The Néron-Severi group NS(X) is a finitely generated abelian group and the *Picard number* of X is defined to be

$$\rho(X) := \mathrm{rank\ NS(X)}.$$

Our main result Theorem 2.1 gives lower bounds for $\varepsilon(X, L, r)$ when X has Picard number 1 and L is the ample generator of NS(X). Some examples of surfaces with Picard number 1 include $\mathbb{P}^2$, general K3 surfaces, and general hypersurfaces in $\mathbb{P}^3$ of degree at least 4.

Our bounds are not always easily comparable to bounds given previously in the literature, but in many cases, we get better bounds. We give some examples comparing our result with known bounds.

We work throughout over the complex number field $\mathbb{C}$. By a *surface*, we mean a nonsingular projective variety of dimension 2. We denote the Seshadri constant $\varepsilon(X, L, r)$ simply by $\varepsilon(L, r)$ when the surface X is clear from context.

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2. Main theorem

The following is the main theorem of this paper.

Theorem 2.1. *Let X be a smooth projective surface with Picard number 1. Let L be the ample generator of* NS(X)*. Let $r \geq 2$ be an integer. Then we have the following.*

(1) *If* $(r, L^2) = (2,6)$ *then* $\varepsilon(L,r) \geq \frac{3}{2}$. *If the equality holds,* $\varepsilon(L,2)$ *is achieved by a curve* $C \in |L|$ *which passes through two very general points with multiplicity two each.*

(2) *If* $(r, L^2) \neq (2,6)$ *then one of the following holds:*

   (a) $\varepsilon(L,r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$, *or*

   (b) *if* $\varepsilon(L,r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{L^2}{r}}$ *then, for some* $d \geq 1$, *there exists an irreducible, reduced curve* $C \in |dL|$ *which passes through* $s \leq r$ *very general points* $x_1, \ldots, x_s$ *with multiplicity one,* $s - 1 \leq C^2$, *and* $\varepsilon(L,r) = \frac{C\cdot L}{s} = \frac{dL^2}{s}$.

Proof. If $\varepsilon(L,r)$ is optimal $\left(\text{namely, equal to } \sqrt{\frac{L^2}{r}}\right)$, there is nothing to prove. So we assume that $\varepsilon(L,r) < \sqrt{\frac{L^2}{r}}$. In this case, there is in fact an irreducible and reducible curve C which computes $\varepsilon(L,r)$. See [19, Proposition 4.5], [10, Lemma 2.1.2], or [3, Proposition 1.1].

Since the Picard number of X is 1, C is algebraically equivalent, and hence numerically equivalent, to dL for some $d \geq 1$. Let $k = L^2$. Then $L \cdot C = kd$ and $C^2 = kd^2$.

Let $x_1, \ldots, x_r$ be very general points of X so that $\varepsilon(L,r) = \varepsilon(L, x_1, \ldots, x_r)$. Denote the multiplicity of C at x_i to be m_i. We rearrange the points so that $m_1 \geq m_2 \geq \ldots \geq m_r$. Let $s \in \{1, \ldots, r\}$ be such that $m_s > 0$ and $m_{s+1} = \ldots = m_r = 0$.

Since the points $x_1, \ldots, x_r$ are very general, there is a non-trivial one-parameter family of irreducible and reduced curves $\{C_t\}_{t\in T}$ parametrized by some smooth curve T and containing points $x_{1,t}, \ldots, x_{r,t} \in C_t$ with $\mathrm{mult}_{x_{i,t}}(C_t) \geq m_i$ for all $1 \leq i \leq r$ and $t \in T$.

By a result of Ein-Lazarsfeld [6] and Xu [22, Lemma 1], we have

$$d^2k = C^2 \geq m_1^2 + \ldots + m_s^2 - m_s. \tag{2.1}$$

First assume that $m_1 = 1$. Then $m_i = 1$ for all $i = 1, \ldots, s$ and $m_{s+1} = \ldots = m_r = 0$. By (2.1), we have $s - 1 \leq C^2$. Since C computes the Seshadri constant, $\varepsilon(L,r) = \frac{C\cdot L}{s} = \frac{dk}{s}$. So we are in case (2b).

We assume now that $m_1 \geq 2$.

The desired inequality in (2a) is $\frac{L\cdot C}{\sum_{i=1}^r m_i} = \frac{dk}{\sum_{i=1}^r m_i} \geq \sqrt{\frac{r+2}{r(r+3)}}\sqrt{k}$. It is equivalent to

$$d^2k \geq \frac{r+2}{r(r+3)}\left(\sum_{i=1}^r m_i\right)^2. \tag{2.2}$$

Note that it suffices to prove

$$d^2k \geq \frac{s+2}{s(s+3)}\left(\sum_{i=1}^r m_i\right)^2. \tag{2.3}$$

Indeed, it is clear that $\frac{s+2}{s(s+3)} \geq \frac{r+2}{r(r+3)}$ for $s \leq r$.

We use the following inequality which is a special case of [8, Lemma 2.3]. Suppose $m_1 \geq 2$, $s \geq 3$ or if $s = 2$ then $(m_1, m_2) \neq (2, 2)$.

$$\frac{(s+3)s}{s+2}\left(\sum_{i=1}^{s} m_i^2 - m_s\right) \geq \left(\sum_{i=1}^{s} m_i\right)^2. \tag{2.4}$$

If $s \geq 3$ then (2.4) is applicable and together with (2.1), it easily gives (2.3). Suppose next that $s = 2$. If $(m_1, m_2) \neq (2, 2)$ we again use (2.4) to obtain (2.3).

If $(m_1, m_2) = (2, 2)$, the right hand side of (2.3) is $\frac{32}{5}$. By (2.1), $d^2k \geq 6$. So (2.3) fails only if $d = 1$ and $k = 6$. So we have case (1) of the theorem.

Finally, if $s = 1$ then we have $d^2k \geq m_1^2 - m_1$ by (2.1). Since $r \geq 2$, we have $m_1^2 - m_1 \geq \frac{r+2}{r(r+3)}m_1^2$ if $m_1 \geq 2$. So (2.2) holds. If $m_1 = 1$ then we are in case (2b).

This completes the proof. □

Remark 2.2. Let X be any surface and L an ample line bundle on X. The Nagata-Biran-Szemberg Conjecture predicts that $\varepsilon(X, L, r) = \sqrt{\frac{L^2}{r}}$ for $r \geq r_0$, for some r_0 depending on X and L (see [19]). Our bound in Theorem 2.1 shows that on a surface X with Picard number 1, $\varepsilon(X, L, r)$ is arbitrarily close to the optimal value of $\sqrt{\frac{L^2}{r}}$ as r tends to ∞, as we explain below.

If Case (2b) of Theorem 2.1 holds then $\varepsilon(X, L, r) = \frac{dk}{s}$ where $k = L^2$ and C is a curve numerically equivalent to dL which passes through $s \leq r$ general points. Since we always have $\varepsilon(X, L, r) \leq \sqrt{\frac{k}{r}}$, we get $d^2 \leq \frac{s^2}{rk}$. On the other hand, for such a curve C to exist, we must have $s < h^0(X, dL)$. By the Riemann-Roch theorem applied asymptotically, the dimension of the space of global sections of dL is bounded by $\frac{d^2L^2}{2} = \frac{d^2k}{2}$. Hence $s \leq \frac{d^2k}{2}$. Putting the above two inequalities together, we get $d^2 \leq s\frac{s}{rk} \leq \frac{d^2k}{2}\frac{s}{rk}$. This in turn gives $2r \leq s$, which is a contradiction.

Hence Case (2b) of Theorem 2.1 does *not* hold for large r. This means that Case (2a) does. So we conclude that $\varepsilon(X, L, r)$ is arbitrarily close to the optimal value of $\sqrt{\frac{L^2}{r}}$ as r tends to infinity.

Remark 2.3. Let X be any surface and L a nef and big line bundle. In [10, Theorem 1.2.1] Harbourne and Roé give lower bounds for $\varepsilon(X, L, r)$. When X has Picard number 1 and L is an ample line bundle on X, [10, Corollary 1.2.2] gives similar bounds. The bounds stated in [10] are implicit and therefore difficult to compare with our bounds. In all situations when the statements in [10] can be made effective, the bounds in [10] turn out to be better than our bounds in Theorem 2.1.

As an application, Harbourne and Roé give good lower bounds when $X = \mathbb{P}^2$; see [10, Corollary 1.2.3]. In order to use these bounds, one needs information about the smallest degree curves passing through r general points with a given multiplicity m. For specific classes of surfaces, it may be possible to obtain this information and in turn get good lower bounds for $\varepsilon(X, L, r)$.

Remark 2.4. Let X be a surface with Picard number 1 and let L be the ample generator of NS(X). Let $r \geq 1$. Szemberg [20, Theorem 3.2] obtains the following bound:

$$\varepsilon(L,r) \geq \left\lfloor \sqrt{\frac{L^2}{r}} \right\rfloor . \tag{2.5}$$

We compare our result Theorem 2.1 with this bound.

If $\frac{L^2}{r}$ is the square of an integer, (2.5) shows that the Seshadri constant is optimal, namely equal to $\sqrt{\frac{L^2}{r}}$, while our bound is sub-optimal. So the bound in (2.5) is always better in this situation.

On the other hand, our bound is always better if $L^2 < r$, because the bound in (2.5) is 0 in this case.

If L^2 is large compared to r, the bound in (2.5) is better. More precisely, for a fixed r, there is some N_r such that if $L^2 \geq N_r$, (2.5) uniformly gives a better bound. However, if $L^2 < N_r$, our bound is better in some cases and (2.5) is better in some cases.

We give an example to illustrate this. Before considering the example, we note that given any integer $k \geq 4$, a general hypersurface X of degree k in $\mathbb{P}^3$ has Picard number 1. In fact, if $L = \mathcal{O}_X(1)$, the Picard group of X is isomorphic to $\mathbb{Z} \cdot L$, by Noether-Lefschetz. Further, $L^2 = k$.

Let $r = 10$. For $L^2 = 150$, the the bound in [20] is 3, while our bound is 3.72. For $L^2 = 1050$, the bound in [20] is 10, while ours is 9.84. If $L^2 = 2500$, then the bound in [20] is 15, and our bound is 15.19. Starting at about $L^2 = 5000$, the bound in [20] is uniformly better than ours.

Remark 2.5. Let X be a surface and let L be a very ample line bundle. Let $k = L^2$. Let $\varepsilon_{r,k}$ be the maximum element in the following set:

$$\left\{ \frac{\lfloor d\sqrt{rk} \rfloor}{dr} \middle| 1 \leq d \leq \sqrt{\frac{r}{k}} \right\} \cup \left\{ \frac{1}{\lceil \sqrt{\frac{r}{k}} \rceil} \right\} \cup \left\{ \frac{dk}{\lceil d\sqrt{rk} \rceil} \middle| 1 \leq d \leq \sqrt{\frac{r}{k}} \right\}.$$

Harbourne [9, Theorem I.1] proves that $\varepsilon(L,r) \geq \varepsilon_{r,k}$, unless $k \leq r$ and rk is a square, in which case $\sqrt{\frac{k}{r}} = \varepsilon_{r,k}$ and $\varepsilon(L,r) \geq \sqrt{\frac{k}{r}} - \delta$ for every positive rational number δ.

Suppose now that X has Picard number 1 and let L be a very ample line bundle on X.

When $r < L^2$ our bound is better than [9, Theorem I.1]. If $r \geq L^2$, our result is not always uniformly comparable with this result. The two bounds are in general very close to each other. We give an example to illustrate this. Let $r = 10$. Then for $L^2 = 6$ our bound is 0.744 and the bound in [9] is 0.75. On the other hand, for $L^2 = 7$, our bound is 0.803 while the bound in [9] is 0.8.

It should be noted however that [9, Theorem I.1] is a general result proved for a very ample line bundle on *any* surface.

Remark 2.6. We compare our bounds with the bounds that can be derived from (1.1).

Let X be a surface with Picard number 1 and let L be the ample generator of NS(X). [20, Conjecture] says that $\varepsilon(X, L, 1) \geq \frac{p_o}{q_o}k$ where $k = L^2$ and (p_o, q_0) is a primitive solution to Pell's equation: $q^2 - kp^2 = 1$. [7] verifies this conjecture whenever k is of the form $n^2 - 1$ or $n^2 + 1$ for a positive integer n. In all cases (not just if X has Picard number one) it is proved that $\varepsilon(X, L, 1) \geq \frac{p_o}{q_o}k$ or $\varepsilon(X, L, 1)$ belongs to a finite set of exceptional values.

Let $r = 101$, $L^2 = 35$. Then our bound is 0.5858. Assuming $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r) = \sqrt{\frac{1}{r}}$ (which is the value predicted by the Nagata Conjecture) and using the bound $\varepsilon(X, L, 1) \geq \frac{35}{6}$ in [7, Theorem 4.1], (1.1) gives us $\varepsilon(X, L, r) \geq 0.5804$. If L is very ample, [9, Theorem I.1] gives the bound 0.5833. Note that the optimal value of $\varepsilon(X, L, r)$ is $\sqrt{\frac{35}{101}} = 0.5886$ in this case.

Bauer [1, Theorem 6.1] explicitly calculates the Seshadri constants $\varepsilon(X, L, 1)$, where X is an abelian surface with Picard number 1, in terms of the primitive solution to Pell's equation. These values, together with bounds on $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r)$ can be used (via (1.1)) to obtain bounds for $\varepsilon(X, L, r)$. These bounds are sometimes better than our bound, but not always.

2.1. **Examples.**

Example 2.7. Let $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.

The values of $\varepsilon = \varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1, r))$ for $2 \leq r \leq 9$ are given below:

$\underline{r = 2, 3, 4}$: Here $\varepsilon = 1/2$. In all three cases the Seshadri constant is achieved by a line through two points. Note that the bound expected in Case (2a) in Theorem 2.1 is 0.63, 0.52 and 0.4 when $r = 2, 3, 4$ respectively. So when $r = 2$ or $r = 3$, Case (2b) in Theorem 2.1 is realized.

$\underline{r = 5}$: In this case $\varepsilon = 2/5$, achieved by a conic through five general points. The bound predicted by Case (2a) in Theorem 2.1 is 0.41 which is more than 2/5. So this is also an instance where Case (2b) in Theorem 2.1 is achieved.

$\underline{r = 6, 7, 8, 9}$: In these cases $\varepsilon = 2/5, 3/8, 6/17, 3$ respectively. In each case, the bound of Case (2a) of Theorem 2.1 holds.

For $r \geq 10$, the Nagata Conjecture says that $\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1, r)) = \frac{1}{\sqrt{r}}$.

Example 2.8. Let X be a general K3 surface. It is well-known that Picard number is 1 for such surfaces. Let L be the ample generator of NS(X) with $k = L^2$. Let $r \geq 3$. We will show in this case that $\varepsilon(X, L, r) \geq \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{k}{r}}$.

If not, by Theorem 2.1, for some $d \geq 1$, there exists a curve $C \in |dL|$ passing through $s \leq r$ very general points with multiplicity one at each point and such that $\varepsilon(X, L, r) = \frac{L \cdot C}{s}$.

We first suppose that $C^2 = d^2k \geq s$. Then $\varepsilon(X, L, r) = \frac{dk}{s} \geq \sqrt{\frac{k}{r}}$. But this violates the assumption that $\varepsilon(X, L, r) < \sqrt{\frac{r+2}{r+3}}\sqrt{\frac{k}{r}}$

Now we compute $h^0(dL)$ and show that $C^2 \geq s$ always holds. By the Kodaira vanishing, $h^1(dL) = h^2(dL) = 0$. It follows from the Riemann-Roch theorem that $h^0(dL) = \frac{d^2k}{2} + 2$ for $d \geq 1$. Since the linear system $|dL|$ contains a curve through s general points, its dimension is at least s. On the other hand, if its dimension is more than s, then $|dL|$ contains a curve D through at least $s+1$ points. Then by inequality (2.1), $d^2k = C^2 = D^2 \geq s$. In this case, we are done by the argument in the previous paragraph. So suppose that $h^0(dL) = \frac{d^2k}{2} + 2 = s+1$. Hence $C^2 = d^2k = 2s-2$. If $s \geq 2$, then $C^2 \geq s$. If $s = 1$, then anyway $C^2 = d^2k \geq 1$. So in both cases, we are done by the argument in the previous paragraph.

Remark 2.9. Another class of surfaces which can have Picard number 1 are abelian surfaces. Bauer [1] studies both single and multi-point Seshadri constants on such surfaces generally (not just when the Picard number is 1). In [1, Section 8], lower bounds for multi-point Seshadri constants are obtained for *arbitrary* points $x_1, \ldots, x_r$. These bounds are naturally smaller than our bounds for $\varepsilon(L, r)$, which is the Seshadri constant at r *general* points.

Remark 2.10. We do not know of any example where Case (1) of Theorem 2.1 is realized.

Though the Nagata-Biran-Szemberg Conjecture (see Remark 2.2) predicts that for a fixed polarized surface (X, L), the Seshadri constants $\varepsilon(X, L, r)$ are optimal for large enough r, we can ask the following question.

Question 2.11. Given an integer $r \geq 1$, is there a pair (X, L) where X is a surface with Picard number 1 and L is the ample generator of NS(X) such that $\varepsilon(X, L, r)$ is sub-optimal, that is, $\varepsilon(X, L, r) < \sqrt{\frac{L^2}{r}}$?

For $r = 1$, an affirmative answer is given by Bauer [1, Theorem 6.1]. This result shows that for a polarized abelian surface (X, L) of type $(1, d)$, the Seshadri constant $\varepsilon(X, L, 1)$ is sub-optimal if $L^2 = 2d$ is not a square. For $r = 2, 3, 5, 6, 7$ and 8, $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ (see Example 2.7) gives an affirmative answer. For other values of r, we do not know an answer to Question 2.11.

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