SESHADRI CONSTANTS AND GRASSMANN BUNDLES OVER CURVES

INDRANIL BISWAS, KRISHNA HANUMANTHU, D. S. NAGARAJ, AND PETER E. NEWSTEAD

ABSTRACT. Let X be a smooth complex projective curve, and let E be a vector bundle on X which is not semistable. For a suitably chosen integer r, let $\mathrm{Gr}(E)$ be the Grassmann bundle over X that parametrizes the quotients of the fibers of E of dimension r. Assuming some numerical conditions on the Harder-Narasimhan filtration of E, we study Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$. In many cases, we give the precise values of the Seshadri constants. Our results generalize various known results for the special case of $\mathrm{rank}(E) = 2$. We include some examples in rank 4.

1. INTRODUCTION

Seshadri constants were introduced by Demailly [Dem] in 1990. He was motivated by an ampleness criterion of Seshadri [Har1, Theorem 7.1]. Seshadri constants have become an important area of research dealing with positivity of line bundles on projective varieties.

Let M be a smooth complex projective variety, and let L be a nef line bundle on M. Take any point $x \in M$. The *Seshadri constant of L at x*, which is a nonnegative real number denoted by $\varepsilon(M, L, x)$, is defined to be

$$\varepsilon(M, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all closed curves in M passing through the point x. Here $L \cdot C$ denotes the intersection multiplicity, so only the numerical class of L is relevant in the above definition, and $\mathrm{mult}_x C$ denotes the multiplicity of the curve C at x. It is easy to check that the infimum can equivalently be taken just over irreducible and reduced curves C.

Seshadri's above mentioned criterion for ampleness says that L is ample if and only if $\varepsilon(M, L, x) > 0$ for all $x \in M$.

If the variety M is clear from the context, we simply write $\varepsilon(L, x)$ instead of $\varepsilon(M, L, x)$.

Usually, the Seshadri constants are not easy to compute precisely and a lot of work has focused on giving bounds on Seshadri constants. If L is an ample line bundle on a projective variety M of dimension n, then it is not difficult to show that $\varepsilon(L, x) \leq \sqrt[n]{L^n}$ for all $x \in M$, where L^n denotes the top self-intersection of L. So $\varepsilon(L, x)$ belongs to the interval $(0, \sqrt[n]{L^n}]$. In many specific cases, it is an interesting problem to shrink this range

of possible values for Seshadri constants and a lot of work has been done in this direction. For an overview of current research on Seshadri constants, see [BDHKKSS].

Since the Seshadri constant of an ample line bundle L at every point x is bounded above by $\sqrt[n]{L^n}$, we can consider their supremum as x varies and define[1]

$$\varepsilon(L,1) := \sup_{x \in M} \varepsilon(L,x).$$

We also define

$$\varepsilon(L) := \inf_{x \in M} \varepsilon(L,x).$$

When L is ample, the following inequalities hold for every $x \in M$:

$$0 < \varepsilon(L) \leq \varepsilon(L,x) \leq \varepsilon(L,1) \leq \sqrt[n]{L^n}.$$

An example of Miranda [La2, Example 5.2.1] shows that $\varepsilon(L)$ can be arbitrarily small; to be more precise, given a real number $\delta > 0$, there exists an algebraic surface M (which is obtained by blowing up the projective plane $\mathbb{P}^2$ at suitably chosen points) and an ample line bundle L on M such that $\varepsilon(M,L) < \delta$.

On the other hand, Ein and Lazarsfeld, [EL], have proved that $\varepsilon(M,L,1) \geq 1$ for any ample line bundle L on any surface M. In fact, they show that the inequality $\varepsilon(M,L,x) < 1$ holds for at most countably many points $x \in M$. In the case of surfaces, it is known that $\varepsilon(M,L,1)$ is achieved at *very general* points on M (that is, points outside a countable union of Zariski closed proper subsets in M); for more details, see [Og].

As the two results mentioned above suggest, most of the research on Seshadri constants is focused on the case of surfaces. There are many important open problems on Seshadri constants for surfaces which attract most attention and not a lot is known for higher dimensional varieties.

Let M be a smooth complex projective variety of dimension n. For any ample line bundle L on M, it is proved in [EKL] that $\varepsilon(L,1) \geq \frac{1}{n}$. This partially generalizes the earlier mentioned result of Ein and Lazarsfeld for $n = 2$. Some improvements for this are known for specific classes of varieties. For example, if M is a threefold, Cascini and Nakamaye in [CN] obtained the bound $\varepsilon(L,1) > \frac{1}{2}$. If M is a Fano variety of dimension $n \geq 3$ and its anti-canonical divisor is globally generated, then $\varepsilon(L) \geq \frac{1}{n-2}$, except when M is a del Pezzo three-fold of degree 1 (see [Le]). When M is a toric variety and more generally when M has toric degenerations, Ito, [It2], has bounds on $\varepsilon(L,1)$ for an ample line bundle L on M. When M is arbitrary, there are bounds on $\varepsilon(L,1)$ in terms of Seshadri constants on the toric variety associated to the Newton–Okounkov body of L [It1]. If M is an abelian variety, we have $\varepsilon(L,x) = \varepsilon(L) = \varepsilon(L,1)$ for all $x \in M$, because a translation does not change Seshadri constants. The case of abelian varieties has been studied extensively; see, for example, [Na, La1, Ba, Deb].

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[1]It may be mentioned that the numeral 1 in the definition of $\varepsilon(L,1)$ refers to the fact that we are considering Seshadri constants at one point. Seshadri constants can also be defined at a finite set of points and in this generality they are called *multi-point* Seshadri constants.

Seshadri constants on ruled surfaces have been looked at by many authors (see [Ga, HM, Sy]). Since ruled surfaces are projective bundles associated to rank two vector bundles over smooth projective curves, it is natural to ask if similar results can be proved for projective bundles of higher rank vector bundles on smooth curves. Our aim here is to study Seshadri constants for line bundles on certain Grassmann bundles over smooth complex projective curves.

Let X be a smooth complex projective curve, and let E be a vector bundle over X of rank n. For a fixed integer $1 \leq r < n$, let $\mathrm{Gr}(E)$ denote the Grassmann bundle over X parametrizing all r-dimensional quotients of fibers of E. We study Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$ for a suitably chosen r. Some numerical assumptions on the Harder-Narasimhan filtration of E are imposed to obtain the results on Seshadri constants.

A related problem has been studied in [BSS], [Hac]. In these papers, the authors define Seshadri constants for a vector bundle E on a projective variety X in terms of the tautological line bundle on the projective bundle $\mathbb{P}(E)$ over X. They also give some bounds on these Seshadri constants. We remark that this is different from directly considering Seshadri constants of line bundles on $\mathbb{P}(E)$.

In Section 2, we investigate the geometry of $\mathrm{Gr}(E)$ and obtain the results required for the computation of Seshadri constants. In Section 3, we study Seshadri constants for ample line bundles L on $\mathrm{Gr}(E)$. We have two main results (Theorems 3.1 and 3.3) on Seshadri constants. They assume different conditions on the Harder-Narasimhan filtration of E. We also prove two corollaries (Corollary 3.2 and Corollary 3.4) to our main theorems which give values of $\varepsilon(L, 1)$ and $\varepsilon(L)$. Our results generalize many of the known results in the case of ruled surfaces (i.e., when E has rank two); this is discussed in Remark 4.1. We also give several examples to illustrate our results.

2. Unstable bundles and real Néron–Severi group

2.1. **Nef and pseudo-effective cones of a Grassmann bundle.** Let X be a connected smooth complex projective curve. The genus of X will be denoted by g. Let E be a vector bundle over X which is not semistable. Let

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{d-1} \subset E_d = E \tag{2.1}$$

be the Harder–Narasimhan filtration of E. Note that $d \geq 2$ because E is assumed not to be semistable.

Fix an integer $1 < m \leq d$. Let

$$r := \mathrm{rank}(E/E_{m-1}).$$

Let

$$f : \mathrm{Gr}(E) \longrightarrow X \tag{2.2}$$

be the Grassmann bundle that parametrizes the quotients of the fibers of E of dimension r. The real Néron–Severi group for $\mathrm{Gr}(E)$ is defined to be

$$\mathrm{NS}(\mathrm{Gr}(E))_{\mathbb{R}} := \mathrm{NS}(\mathrm{Gr}(E)) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathrm{Pic}(\mathrm{Gr}(E))/\mathrm{Pic}^0(\mathrm{Gr}(E))) \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\mathrm{Pic}^0(\mathrm{Gr}(E))$ is the connected component of the Picard group $\mathrm{Pic}(\mathrm{Gr}(E))$ containing the identity element. A class $c \in \mathrm{NS}(\mathrm{Gr}(E))$ is called effective if there is an effective divisor D on $\mathrm{Gr}(E)$ such that c represents $\mathcal{O}_{\mathrm{Gr}(E)}(D)$. The *pseudo-effective cone* of $\mathrm{Gr}(E)$ is the closed cone in $\mathrm{NS}(\mathrm{Gr}(E))_{\mathbb{R}}$ generated by the effective classes. The nef cone of $\mathrm{Gr}(E)$ coincides with the closed cone in $\mathrm{NS}(\mathrm{Gr}(E))_{\mathbb{R}}$ generated by the ample divisors on $\mathrm{Gr}(E)$. The nef cone is contained in the pseudo-effective cone.

For notational convenience, we will write

$$\theta := \mathrm{degree}(E/E_{m-1}). \tag{2.3}$$

Let

$$\mathcal{L} \in \mathrm{NS}(\mathrm{Gr}(E)) \tag{2.4}$$

be the class of a fiber of the projection f in (2.2), so $\mathcal{L}$ is the class of f^*L_1 for a line bundle L_1 on X of degree one. The tautological line bundle on $\mathrm{Gr}(E)$ (the r-th exterior product of the tautological quotient vector bundle) will be denoted by $\mathcal{O}_{\mathrm{Gr}(E)}(1)$.

We will now recall Proposition 4.1 of page 363 in [BP] since it will be used here a couple of times. Take any integer $1 \leq r'' < \mathrm{rank}(E)$. Let $1 \leq t \leq d$ be the unique largest integer such that

$$\mathrm{rank}(E/E_{t-1}) \geq r''$$

(see (2.1)).

For any vector bundle V on X, denote $\mathrm{degree}(V)/\mathrm{rank}(V) \in \mathbb{Q}$ by $\mu(V)$.

Define

$$\theta_{E,r''} := (r'' - \mathrm{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1}) + \mathrm{degree}(E/E_t). \tag{2.5}$$

Let $\varphi : \mathrm{Gr}_{r''}(E) \longrightarrow X$ be the Grassmann bundle parametrizing the r'' dimensional quotients of the fibers of E. Let $\mathcal{O}_{\mathrm{Gr}_{r''}(E)}(1) \longrightarrow \mathrm{Gr}_{r''}(E)$ be the determinant of the tautological bundle over $\mathrm{Gr}_{r''}(E)$ of rank r''. Just as in (2.4), let

$$\mathcal{L}'' \in \mathrm{NS}(\mathrm{Gr}_{r''}(E))$$

be the class of a fiber of the above projection φ.

Proposition 2.1 ([BP, Proposition 4.1]). *The nef cone of $\mathrm{Gr}_{r''}(E)$ is generated by $\mathcal{L}''$ and $[\mathcal{O}_{\mathrm{Gr}_{r''}(E)}(1)] - \theta_{E,r''} \cdot \mathcal{L}''$, where $\mathcal{L}''$ is defined above.*

If there is a positive integer $c < d$ such that $\mathrm{rank}(E_c) = r$ (see (2.1)), then define

$$\zeta := \mathrm{degree}(E_c). \tag{2.6}$$

Lemma 2.2.

(1) *The nef cone of $\mathrm{Gr}(E)$ is generated by $\mathcal{L}$ (defined in (2.4)) and $\mathcal{M} := [\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \theta \cdot \mathcal{L}$ (see (2.3)).*

(2) *Assume that there is a positive integer $c < d$ such that $\mathrm{rank}(E_c) = r$. Then the pseudo-effective cone of $\mathrm{Gr}(E)$ is generated by $\mathcal{L}$ and $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$, where ζ is defined in (2.6).*

Proof. The first statement is a particular case of Proposition 2.1.

The second statement is proved in [BHP] (see [BHP, p. 74, Theorem 4.1]). Note that the rational number $\zeta_{E,r}$, defined in [BHP, p. 74, (3.10)], coincides with our ζ. $\qquad\square$

Proposition 2.3. *Assume that ζ in (2.6) is an integral multiple of r. Then the class $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L} \in \mathrm{NS}(\mathrm{Gr}(E))$ is effective. Furthermore, there is exactly one effective divisor on $\mathrm{Gr}(E)$ whose class is $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$.*

Proof. Let L be a line bundle on X such that $L^{\otimes r} = \bigwedge^r (E_c)^*$ (recall that ζ is an integral multiple of r). We note that $\mathrm{degree}(E_c \otimes L) = 0$, in fact the determinant line bundle $\bigwedge^r (E_c \otimes L)$ is the trivial line bundle.

Replace E by $\mathcal{E} := E \otimes L$. Let

$$h : \mathrm{Gr}(\mathcal{E}) \longrightarrow X \tag{2.7}$$

be the Grassmann bundle that parametrizes the quotients of the fibers of $\mathcal{E}$ of dimension r. Then $\mathrm{Gr}(\mathcal{E})$ is canonically identified with $\mathrm{Gr}(E)$.

Given a vector bundle U' and a line bundle L' on X, the vector bundle $U' \otimes L'$ is semistable if and only if U' is semistable. Also, we have $\mu(U' \otimes L') = \mu(U') + \mu(L')$. Therefore, the Harder–Narasimhan filtration of $U' \otimes L'$ is simply the Harder–Narasimhan filtration of U' tensored with the line bundle L'. In particular, the Harder–Narasimhan filtration of $\mathcal{E}$ is the filtration in (2.1) tensored with L, meaning the filtration

$$0 = E_0 \otimes L \subset E_1 \otimes L \subset \cdots \subset E_{d-1} \otimes L \subset E_d \otimes L = E \otimes L = \mathcal{E}$$

is the Harder–Narasimhan filtration of $\mathcal{E}$. Since $\mathrm{degree}(E_c \otimes L) = 0$, the class of $\mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1)$ in $\mathrm{NS}(\mathrm{Gr}(E)) = \mathrm{NS}(\mathrm{Gr}(\mathcal{E}))$ coincides with the class $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$ in Lemma 2.2(2).

We have

$$\bigwedge^r \mathcal{E} \supset \bigwedge^r (E_c \otimes L) = (\bigwedge^r E_c) \otimes L^{\otimes r} = \mathcal{O}_X . \tag{2.8}$$

This inclusion of the coherent sheaf $\mathcal{O}_X$ in $\bigwedge^r \mathcal{E}$ produces an inclusion

$$\mathbb{C} = H^0(X, \mathcal{O}_X) \hookrightarrow H^0(X, \bigwedge^r \mathcal{E}) = H^0(\mathrm{Gr}(\mathcal{E}), \mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1)) .$$

Let D be the effective divisor on $\mathrm{Gr}(E) = \mathrm{Gr}(\mathcal{E})$ given by this one-parameter family of nonzero sections.

Since the class of $\mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1)$ in $\mathrm{NS}(\mathrm{Gr}(E))$ is $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$, the effective divisor D constructed above lies in the class $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$. In particular, the class $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L}$ is effective.

Now we will show that there is only one effective divisor in the class $[\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \zeta \cdot \mathcal{L} = [\mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1)]$.

Let D' be an effective divisor on $\mathrm{Gr}(\mathcal{E})$ that lies in the class $[\mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1)]$. Then

$$\mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(D') = \mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1) \otimes h^* L_0 , \tag{2.9}$$

where L_0 is a line bundle on X of degree zero and h is the projection to X in (2.7). Now using the projection formula, we have

$$H^0(\mathrm{Gr}(\mathcal{E}), \mathcal{O}_{\mathrm{Gr}(\mathcal{E})}(1) \otimes h^* L_0) = H^0(X, (\bigwedge^r \mathcal{E}) \otimes L_0) .$$

Let U be a vector bundle on X, let $U_1 \subset U$ be the first nonzero term in the Harder–Narasimhan filtration for U. For any integer $1 \leq s \leq \mathrm{rank}(U_1)$, the first nonzero term in the Harder–Narasimhan filtration for the vector bundle $\bigwedge^s U$ is $\bigwedge^s U_1$. Furthermore, if U' is any nonzero term in the Harder–Narasimhan filtration for U, then the first nonzero term in the Harder–Narasimhan filtration for $\bigwedge^b U$, where $b = \mathrm{rank}(U')$, is $\bigwedge^b U'$.

Since $r = \mathrm{rank}(E_c \otimes L)$, from the above property of the Harder–Narasimhan filtrations we know that the first nonzero term in the Harder–Narasimhan filtration for $\bigwedge^r \mathcal{E}$ is the line bundle $\bigwedge^r (E_c \otimes L) = (\bigwedge^r E_c) \otimes L^{\otimes r} = \mathcal{O}_X$ (as in (2.8)). Therefore, the first nonzero term in the Harder–Narasimhan filtration for $(\bigwedge^r \mathcal{E}) \otimes L_0$ is the line bundle $\bigwedge^r (E_c \otimes L) \otimes L_0 = L_0$ (recall that the Harder–Narasimhan filtration of the tensor product of U with a line bundle is the tensor product of the Harder–Narasimhan filtration U with the line bundle). We have $\mathrm{degree}(L_0) = 0$, so all the other graded pieces of the Harder–Narasimhan filtration for $(\bigwedge^r \mathcal{E}) \otimes L_0$ (other than L_0) are of negative degree. Since a semistable vector bundle of negative degree does not admit any nonzero section, from the Harder–Narasimhan filtration for $(\bigwedge^r \mathcal{E}) \otimes L_0$ it follows that

$$H^0(X, (\bigwedge^r \mathcal{E}) \otimes L_0) \subset H^0(X, L_0). \tag{2.10}$$

On the other hand, from (2.9) it follows that

$$H^0(X, (\bigwedge^r \mathcal{E}) \otimes L_0) \neq 0.$$

Since $\mathrm{degree}(L_0) = 0$, from (2.10) it now follows that

- L_0 is the trivial line bundle $\mathcal{O}_X$, and
- $\dim H^0(X, (\bigwedge^r \mathcal{E}) \otimes L_0) = 1$.

This completes the proof of the proposition. $\qquad\square$

2.2. **Bound on the degree of a quotient.** Recall that m is a fixed integer with $1 < m \leq d$, and $r = \mathrm{rank}(E/E_{m-1})$. The following is a corollary of Lemma 2.2(1):

Corollary 2.4. *Let W be a quotient bundle of E of rank r. Then $\mathrm{degree}(W) \geq \theta$.*

Proof. Let

$$s_W : X \longrightarrow \mathrm{Gr}(E)$$

be the section of the projection f in (2.2) defined by W. For the class $\mathcal{M}$ in Lemma 2.2(1), we have

$$s_W^*(\mathcal{M}) = s_W^*([\mathcal{O}_{\mathrm{Gr}(E)}(1)]) - \theta \geq 0$$

($\mathrm{NS}(X)$ is identified with $\mathbb{Z}$ using degree), because $\mathcal{M}$ is nef by Lemma 2.2(1) and $s_W^*(\mathcal{L}) = 1$. But

$$s_W^*([\mathcal{O}_{\mathrm{Gr}(E)}(1)]) = \mathrm{degree}(W),$$

hence we have $\mathrm{degree}(W) \geq \theta$. $\qquad\square$

The following is a refinement of Corollary 2.4:

Proposition 2.5. *Let W be a quotient bundle of E of rank r. Then exactly one of the following two is valid:*

(1) *The quotient W coincides with the quotient E/E_{m-1} in (2.1).*

(2) $\text{degree}(W) \geq \theta + \mu(E_{m-1}/E_{m-2}) - \mu(E_m/E_{m-1})$.

Proof. Since $\mu(E_m/E_{m-1}) < \mu(E_{m-1}/E_{m-2})$, if statement (2) in the proposition holds, then statement (1) is not valid. So it suffices to prove that (2) holds assuming that (1) does not hold.

Consider the quotient bundles W and E/E_{m-1} of E of rank r. Let S be the image in W of the subbundle $E_{m-1} \subset E$. Therefore,

$$Q := W/S$$

is a quotient of E/E_{m-1}. So, we have

$$\text{degree}(W) = \text{degree}(S) + \text{degree}(Q). \tag{2.11}$$

Assume that W is not the quotient E/E_{m-1} of E. This implies that we have $S \neq 0$ and $\text{rank}(Q) < r$. Let r' be the rank of Q; so $\text{rank}(S) = r - r' > 0$.

Consider the Harder–Narasimhan filtration of E in (2.1). Let t be the smallest positive integer such that $\text{rank}(E/E_t) < r'$. Note that $t > m - 1$ because $r' < r$. Let

$$q_0 : \text{Gr}_{r'}(E/E_{m-1}) \longrightarrow X$$

be the Grassmann bundle that parametrizes all the r' dimensional quotients of the fibers of E/E_{m-1}. The real Néron–Severi class

$$[\mathcal{O}_{\text{Gr}_{r'}(E/E_{m-1})}(1)] - (\text{degree}(E/E_t) + (r' - \text{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1})) \cdot q_0^* L_1$$

on $\text{Gr}_{r'}(E/E_{m-1})$ is nef (see Proposition 2.1), where $\mathcal{O}_{\text{Gr}_{r'}(E/E_{m-1})}(1)$ is the tautological line bundle, and L_1 is the class of a line bundle on X of degree 1. Consequently, evaluating this class on the section of the projection q_0 given by the torsionfree quotient of Q we have

$$\text{degree}(Q) \geq \text{degree}(Q/\text{Torsion}) \geq \text{degree}(E/E_t) + (r' - \text{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1}). \tag{2.12}$$

We will next prove that

$$\text{degree}(S) \geq (r - r') \cdot \mu(E_{m-1}/E_{m-2}). \tag{2.13}$$

For this, first note that if the natural surjective homomorphism $E_{m-1} \longrightarrow S$ is also injective, then (2.13) holds, because in that case,

$$\text{degree}(E_{m-1}) \geq \text{rank}(E_{m-1}) \cdot \mu(E_{m-1}/E_{m-2}) = (r - r') \cdot \mu(E_{m-1}/E_{m-2}).$$

So to prove (2.13) we will assume that $r - r' < \text{rank}(E_{m-1})$.

Let $u \leq m - 1$ be the smallest nonnegative integer such that $0 \leq \text{rank}(E_{m-1}) - \text{rank}(E_u) < r - r'$; note that $u \geq 1$, because $r - r' < \text{rank}(E_{m-1})$. Let

$$q_1 : \text{Gr}_{r-r'}(E_{m-1}) \longrightarrow X$$

be the Grassmann bundle that parametrizes all the $r - r'$ dimensional quotients of the fibers of E_{m-1}.

By Proposition 2.1, the line bundle

$$[\mathcal{O}_{\mathrm{Gr}_{r-r'}(E_{m-1})}(1)] - (\mathrm{degree}(E_{m-1}/E_u) + (r-r'-\mathrm{rank}(E_{m-1}) + \mathrm{rank}(E_u)) \cdot \mu(E_u/E_{u-1})) \cdot q_1^* L_1$$

on $\mathrm{Gr}_{r-r'}(E_{m-1})$ is nef. Evaluating this line bundle on the section of q_1 given by S (note that S is a locally free quotient of E_{m-1} of rank $r - r'$), it follows that

$$
\begin{aligned}
\mathrm{degree}(S) \quad &\geq\quad \mathrm{degree}(E_{m-1}/E_u) + (r - r' - \mathrm{rank}(E_{m-1}) + \mathrm{rank}(E_u)) \cdot \mu(E_u/E_{u-1}) \\
&=\quad \mathrm{rank}(E_{m-1}/E_u) \cdot \mu(E_{m-1}/E_u) \qquad\qquad\qquad\qquad\qquad (2.14) \\
&\qquad + (r - r' - \mathrm{rank}(E_{m-1}) + \mathrm{rank}(E_u)) \cdot \mu(E_u/E_{u-1}) \\
&\geq\quad (\mathrm{rank}(E_{m-1}) - \mathrm{rank}(E_u)) \cdot \mu(E_{m-1}/E_{m-2}) \qquad\qquad\quad (2.15) \\
&\qquad + (r - r' - \mathrm{rank}(E_{m-1}) + \mathrm{rank}(E_u)) \cdot \mu(E_{m-1}/E_{m-2}). \\
&=\quad (r - r') \cdot \mu(E_{m-1}/E_{m-2}).
\end{aligned}
$$

The inequality (2.15) above holds because

$$\mu(E_{m-1}/E_{m-2}) \leq \mu(E_{m-1}/E_u) \text{ and } \mu(E_{m-1}/E_{m-2}) \leq \mu(E_u/E_{u-1}).$$

Note that the first inequality above holds if $u < m - 1$. If $u = m - 1$ then we directly get (2.13) from (2.14).

This completes the proof of the inequality (2.13).

From (2.11), (2.12) and (2.13) we have

$$
\begin{aligned}
\mathrm{degree}(W) &\geq \mathrm{degree}(E/E_t) + (r' - \mathrm{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1}) + (r - r') \cdot \mu(E_{m-1}/E_{m-2}) \\
&= \mathrm{degree}(E/E_t) + (r' - \mathrm{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1}) + (r - r') \cdot \mu(E_m/E_{m-1}) \\
&\qquad + (r - r') \cdot (\mu(E_{m-1}/E_{m-2}) - \mu(E_m/E_{m-1})). \qquad\qquad (2.16)
\end{aligned}
$$

We will show that

$$\theta \leq \mathrm{degree}(E/E_t) + (r' - \mathrm{rank}(E/E_t)) \cdot \mu(E_t/E_{t-1}) + (r - r') \cdot \mu(E_m/E_{m-1}). \quad (2.17)$$

If $t = m$, then the two sides of (2.17) coincide, because $(r' - \mathrm{rank}(E/E_t)) + (r - r') = \mathrm{rank}(E_m/E_{m-1})$. If $t > m$, then

$$\theta = \mathrm{degree}(E/E_t) + \mathrm{degree}(E_t/E_{t-1}) + \mathrm{degree}(E_{t-1}/E_{m-1})$$

$$= \mathrm{degree}(E/E_t) + (r' - \mathrm{rank}(E/E_t))\mu(E_t/E_{t-1}) + (r - r') \cdot \mu(E_{t-1}/E_{m-1})$$

$$+ (\mathrm{rank}(E/E_{t-1}) - r')\mu(E_t/E_{t-1}) + (r' - r + \mathrm{rank}(E_{t-1}/E_{m-1}))\mu(E_{t-1}/E_{m-1});$$

also, $(\mathrm{rank}(E/E_{t-1}) - r') = -(r' - r + \mathrm{rank}(E_{t-1}/E_{m-1})) > 0$, and $\mu(E_t/E_{t-1}) < \mu(E_{t-1}/E_{m-1})$, so

$$(\mathrm{rank}(E/E_{t-1}) - r')\mu(E_t/E_{t-1}) + (r' - r + \mathrm{rank}(E_{t-1}/E_{m-1}))\mu(E_{t-1}/E_{m-1}) < 0,$$

proving (2.17).

Combining (2.16) and (2.17),

$$\mathrm{degree}(W) \geq \theta + (r - r') \cdot (\mu(E_{m-1}/E_{m-2}) - \mu(E_m/E_{m-1}))$$

$$\geq \theta + \mu(E_{m-1}/E_{m-2}) - \mu(E_m/E_{m-1})$$

because $r - r' \geq 1$ and $\mu(E_{m-1}/E_{m-2}) > \mu(E_m/E_{m-1})$. This completes the proof. $\square$

Write θ in (2.3) as

$$\theta = s_0 + \alpha \cdot r \,,$$

where $\alpha \in \mathbb{Z}$ and $-r \leq s_0 < 0$. Let L be a line bundle on X of degree $-\alpha$ and consider $F := E \otimes L$. Note that $\theta_{F,r} = s_0$ (see (2.5)).

As mentioned before, given a vector bundle U' and a line bundle L' on X, the Harder–Narasimhan filtration of $U' \otimes L'$ is the Harder–Narasimhan filtration of U' tensored with the line bundle L'. In particular, the Harder–Narasimhan filtration for F is the filtration in (2.1) tensored with L.

As noted before, the Grassmann bundles for E and F are identified.

Therefore, substituting E by $E \otimes L$ we can assume that $-r \leq \theta < 0$. Henceforth without loss of generality it may be assumed that

$$-r \leq \theta < 0 \,. \tag{2.18}$$

Let Y be an irreducible smooth complex projective curve and

$$\phi : Y \longrightarrow X$$

a surjective morphism.

Corollary 2.6. *Assume that*

$$\mu(E_m/E_{m-1}) - \mu(E_{m-1}/E_{m-2}) \leq \theta < 0 \,. \tag{2.19}$$

Take any quotient bundle $Q \longrightarrow Y$ of rank r of $\phi^* E$. Then exactly one of the following is valid:

(1) *The quotient Q coincides with the quotient $\phi^*(E/E_{m-1}) = (\phi^* E)/(\phi^* E_{m-1})$.*
(2) $\mathrm{degree}(Q) \geq 0$.

Proof. If U is a semistable vector bundle on X, then the vector bundle $\phi^* U$ on Y is also semistable [BS, p. 441, Theorem 2.4]. (Note that this statement for vector bundles on curves is much easier to prove than the version in [BS] which is for all dimensions.) Also, we have

$$\mu(\phi^* U) = \mathrm{degree}(\phi) \cdot \mu^*(U) \,.$$

From these two facts it follows immediately that the Harder–Narasimhan filtration of $\phi^* U$ is the pullback, via ϕ, of the Harder–Narasimhan filtration of U. In particular, the Harder–Narasimhan filtration of $\phi^* E$ is the pullback, via ϕ, of the filtration in (2.1). Therefore, the result follows from Proposition 2.5 and the conditions on θ in (2.19). $\square$

3. SESHADRI CONSTANTS

In this section, we will use the results in Section 2 to compute Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$.

We quickly recall our set-up. Let X be a smooth complex projective curve, and let E be a vector bundle on X of rank n which is not semistable. We have its Harder-Narasimhan filtration in (2.1). As in Section 2, we fix an integer $1 < m \leq d$ and set

$r := \mathrm{rank}(E/E_{m-1})$. Let $f : \mathrm{Gr}(E) \longrightarrow X$ be the Grassmann bundle that parametrizes the quotients of the fibers of E of dimension r.

In order to compute the Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$, we will first describe its closed cone of curves $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$ (the dual of the nef cone of $\mathrm{Gr}(E)$).

Let Γ_l denote the class of a line in a fiber $Y := f^{-1}(x_0)$. Note that Y is isomorphic to the Grassmannian variety of r-dimensional quotients of an n-dimensional vector space. Let Γ_s denote the image of the section

$$s : X \longrightarrow \mathrm{Gr}(E)$$

corresponding to the rank r quotient $E \longrightarrow E/E_{m-1}$.

We claim that $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$ is the cone spanned by the classes of Γ_s and Γ_l. This can be deduced from the facts that $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$ is the dual cone to the nef cone of $\mathrm{Gr}(E)$, and the nef cone of $\mathrm{Gr}(E)$ is generated by $\mathcal{L}$ and $\mathcal{M}$ by Lemma 2.2(1). Indeed, it is easy to see that $\mathcal{L} \cdot \Gamma_l = 0$ and $\mathcal{L} \cdot \Gamma_s = 1$. On the other hand, $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot \Gamma_s = \theta$, since by definition, $\theta = \mathrm{degree}(E/E_{m-1})$ (see (2.3)). It is also clear that $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot \Gamma_l = 1$. Now recall that $\mathcal{M} = [\mathcal{O}_{\mathrm{Gr}(E)}(1)] - \theta\mathcal{L}$. This gives that $\mathcal{M} \cdot \Gamma_s = 0$ and $\mathcal{M} \cdot \Gamma_l = 1$. It now follows that $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$ is the cone spanned by the classes of Γ_s and Γ_l, proving the claim.

We will now prove two independent results on Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$. These results make different assumptions on the Harder-Narasimhan filtration (2.1) of E.

Theorem 3.1. *Assume that there is an integer $c < d$ such that $\mathrm{rank}(E_c) = r$ and that $\mathrm{degree}(E_c)$ is an integral multiple of r. Let L be an ample line bundle on $\mathrm{Gr}(E)$ which is numerically equivalent to $\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}$, where $\mathcal{L}$ and $\mathcal{M}$ are as in Lemma 2.2(1). Then the Seshadri constants of L are given by the following:*

(1) $\varepsilon(L, x) \geq \min\{a, b\}$, *for all $x \in \mathrm{Gr}(E)$.*
(2) *If $b \leq a$, then $\varepsilon(L, x) = b$, for all $x \in \mathrm{Gr}(E)$.*
(3) *If $a < b$, then:*
    (i) *if x does not belong to the base locus of the linear system $|\mathcal{O}_{Gr(E)}(1)|$, then $\varepsilon(L, x) = b$;*
    (ii) *if x belongs to the base locus of $|\mathcal{O}_{Gr(E)}(1)|$, then $a \leq \varepsilon(L, x) \leq b$;*
    (iii) *if $x \in \Gamma_s$, then $\varepsilon(L, x) = a$.*

Proof. We first prove statements (1) and (2), which, in fact, hold in general, without the assumption on E_c.

Let $C \subset \mathrm{Gr}(E)$ be an irreducible and reduced curve passing through x. Let m denote the multiplicity of C at x. We consider two cases.

Case 1: C is contained in a fiber of f.

In this case, as an element of $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$, this C is given by $n_l \cdot \Gamma_l$ for a positive integer n_l. In other words, C is a curve of degree n_l in a fiber of f which is isomorphic to

the Grassmannian of r-dimensional quotients of an n-dimensional vector space. Since C contains a point x of multiplicity m, its degree is at least m, i.e., $n_l \geq m$.

We then have $\frac{L \cdot C}{m} = \frac{b n_l}{m} \geq b$.

Case 2: C is not contained in a fiber of f.

In this case, as an element of $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$, this C is given by $n_l \Gamma_l + n_s \Gamma_s$, for non-negative integers n_s, n_l. Let Y be the fiber of f passing through x. Since $C \not\subset Y$, Bézout's theorem gives us the inequality

$$Y \cdot C = \mathcal{L} \cdot C \geq \mathrm{mult}_x C = m\,.$$

So we have $n_s \geq m$.

Now the Seshadri ratio is given by $\frac{L \cdot C}{m} = \frac{a n_s + b n_l}{m} \geq a + \frac{b n_l}{m} \geq a$.

By Cases 1 and 2, we have that $\frac{L \cdot C}{m} \geq \min\{a, b\}$, for all $x \in \mathrm{Gr}(E)$. This proves (1). Further, note that for every point $x \in \mathrm{Gr}(E)$, there exists a line l in a fiber of f that contains x. The Seshadri ratio for l is $\frac{L \cdot l}{1} = b$. This means that $\varepsilon(L, x) \leq b$ for all $x \in \mathrm{Gr}(E)$. So if $b \leq a$, then $\varepsilon(L, x) = b$, giving (2).

Now we use the hypothesis of the theorem to prove (3). Because of our hypothesis, we can apply Proposition 2.3. We normalize E as in the proof of Proposition 2.3. In other words, we replace E by $E \otimes L$, where L is a line bundle on X such that $L^{\otimes r} = \bigwedge^r (E_c)^*$ (recall that degree(E_c) is a multiple of r, by hypothesis). Then, as argued in the proof of Proposition 2.3, it follows that $\bigwedge^r E_c = \mathcal{O}_X$. So $\zeta := \mathrm{degree}(E_c) = 0$ and $\theta = \mathrm{degree}(E/E_{m-1}) < 0$. Further, by Proposition 2.3, $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ is effective.

Now we prove (3)(i). So suppose that x is not in the base locus of the linear system $|\mathcal{O}_{\mathrm{Gr}(E)}(1)|$. Then C is also not contained in the base locus of $|\mathcal{O}_{\mathrm{Gr}(E)}(1)|$. This implies that $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot C \geq 0$. Suppose that as an element of $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$, the curve C is given by $n_l \Gamma_l + n_s \Gamma_s$, where n_s, n_l are non-negative integers. Then from the observation that $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot C \geq 0$ it follows that $n_l \geq n_s(-\theta) \geq n_s$.

If C is contained in a fiber of f, then by Case 1, we have $\frac{L \cdot C}{m} \geq b$. If C is not contained in a fiber of f, then we have $n_s \geq m$. So $n_l \geq n_s \geq m$. Hence $\frac{L \cdot C}{m} = \frac{a n_s + b n_l}{m} \geq a + b \geq b$. This completes the proof of (3)(i).

For (3)(ii), we have $\varepsilon(L, x) \geq a$, by part (1) of the theorem. Also, as noted above, for every point $x \in \mathrm{Gr}(E)$, there exists a line l in a fiber of f that contains x whose Seshadri ratio is given by $\frac{L \cdot l}{1} = b$. This means that $\varepsilon(L, x) \leq b$ for all $x \in \mathrm{Gr}(E)$. So (3)(ii) follows.

For (3)(iii), note that Γ_s is smooth and $L \cdot \Gamma_s = a$. $\qquad\qquad\square$

Corollary 3.2. *Assume the hypotheses of Theorem 3.1 hold. Let $L = \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}$ be an ample line bundle on $\mathrm{Gr}(E)$, where $\mathcal{L}$ and $\mathcal{M}$ are as in Lemma 2.2(1). Then we have*

   (1) $\varepsilon(L, 1) = b$, *and*
   (2) $\varepsilon(L) = \min\{a, b\}$.

Proof. Recall that $\varepsilon(L, 1)$ is the supremum of $\varepsilon(L, x)$ as x varies in X, while $\varepsilon(L)$ is the infimum of $\varepsilon(L, x)$ as x varies in X (see Section 1).

If $b \leq a$, then Theorem 3.1 implies that $\varepsilon(L, x) = b$ for all x. So we have $\varepsilon(L, 1) = \varepsilon(L) = b$. On the other hand, if $b > a$, then $\varepsilon(L, x) = b$ for a point x outside the base locus of $|\mathcal{O}_{\mathrm{Gr}(E)}(1)|$ and $a \leq \varepsilon(L, x) \leq b$ if x is in the base locus of $|\mathcal{O}_{\mathrm{Gr}(E)}(1)|$. Note also that $\varepsilon(L, x) = a$ if $x \in \Gamma_s$. So the corollary follows. $\square$

For our next result about Seshadri constants, we will make use of Corollary 2.6. We also normalize E as in (2.18).

Theorem 3.3. *Assume that $\mu(E_m/E_{m-1}) - \mu(E_{m-1}/E_{m-2}) \leq \theta$. Let L be an ample line bundle on $\mathrm{Gr}(E)$ which is numerically equivalent to $\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}$, where $\mathcal{L}$ and $\mathcal{M}$ are as in Lemma 2.2(1). The Seshadri constant of L at $x \in \mathrm{Gr}(E)$ is given by the following:*

$$\varepsilon(L, x) = \begin{cases} b & \text{if } b \leq a \text{ or } x \notin \Gamma_s, \\ a & \text{if } a < b \text{ and } x \in \Gamma_s. \end{cases}$$

Proof. We normalize E as in (2.18). Then $\theta = \mathrm{degree}(E/E_{m-1})$ is negative.

Note that, if C is an irreducible curve in $\mathrm{Gr}(E)$, not contained in a fiber of f, then the restriction of f to C defines a surjective morphism $\phi : Y \to X$, where Y is the normalization of C. Moreover, the embedding of C in $\mathrm{Gr}(E)$ defines a rank r quotient Q of $\phi^\star(E)$. Conversely, for any surjective morphism $\phi : Y \to X$ with Y smooth and irreducible, a rank r quotient of $\phi^*(E)$ defines a morphism $Y \to \mathrm{Gr}(E)$ and hence a curve C in $\mathrm{Gr}(E)$, not contained in a fiber of f. In particular, the section Γ_s, by definition, is given by the rank r quotient $E \longrightarrow E/E_{m-1}$.

Since $\mu(E_m/E_{m-1}) - \mu(E_{m-1}/E_{m-2}) \leq \theta$ by hypothesis, we can apply Corollary 2.6. This corollary says that if C is a curve on $\mathrm{Gr}(E)$ given by a rank r quotient $\phi^*(E) \to Q$, then either Q coincides with the quotient $E \longrightarrow E/E_{m-1}$ or $\mathrm{degree}(Q) \geq 0$. In other words, either $C = \Gamma_s$ or $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot C \geq 0$.

Let $C \subset \mathrm{Gr}(E)$ be an irreducible and reduced curve passing through x. Let m denote the multiplicity of C at x. We consider three different cases.

Case 1: C is contained in a fiber of f.

In this case, we have $\frac{L \cdot C}{m} \geq b$. The proof is exactly the same as in Case 1 in the proof of Theorem 3.1.

Case 2: $C = \Gamma_s$.

Note that Γ_s is a smooth curve, so that $m = 1$. Hence the Seshadri ratio is given by $\frac{L \cdot \Gamma_s}{1} = a$.

Case 3: $C \neq \Gamma_s$ and C is not contained in a fiber of f.

By Corollary 2.6, we have $\mathcal{O}_{\mathrm{Gr}(E)}(1) \cdot C \geq 0$.

As before, as an element of $\overline{\mathrm{NE}}(\mathrm{Gr}(E))$, the curve C is given by $n_l\Gamma_l + n_s\Gamma_s$, for non-negative integers n_s, n_l. The same argument as for Case 2 in the proof of Theorem 3.1 now gives $n_l \geq n_s(-\theta) \geq n_s \geq m$, and hence $\frac{L\cdot C}{m} = \frac{an_s + bn_l}{m} \geq a + b \geq b$.

The above analysis shows that if $C \neq \Gamma_s$, then the Seshadri ratio $\frac{L\cdot C}{\mathrm{mult}_x C}$ is at least b. Note that given any point $x \in \mathrm{Gr}(E)$, there exists a line l passing through x lying in the fiber containing x. Since l is smooth and it is given by Γ_l in the cone of curves of $\mathrm{Gr}(E)$, the Seshadri ratio is $\frac{L\cdot l}{1} = L \cdot \Gamma_l = b$. Thus the Seshadri constant of L at x is equal to b, except when $a < b$ and $x \in \Gamma_s$, in which case $\varepsilon(L, x) = a$.

This completes the proof of the theorem. $\qquad\square$

Corollary 3.4. *Assume the hypotheses of Theorem 3.3 hold. Let $L = \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}$ be an ample line bundle on $\mathrm{Gr}(E)$, where $\mathcal{L}$ and $\mathcal{M}$ are as in Lemma 2.2(1). Then we have*

(1) $\varepsilon(L, 1) = b$, *and*
(2) $\varepsilon(L) = \min\{a, b\}$.

Proof. The proof is very similar to the proof of Corollary 3.2 and follows easily from Theorem 3.3. $\qquad\square$

4. Remarks and Examples

Remark 4.1. As mentioned in Section 1, several authors have studied Seshadri constants on $\mathrm{Gr}(E)$ when E has rank 2 (see [Ga, Sy]). In this remark, we compare our result in Theorem 3.3 with known results in this case.

Let X be a smooth complex projective curve and let E be a rank 2 vector bundle on X. Assume that E is normalized in the sense of [Har2, Chapter V, Section 2]. In other words, $H^0(X, E) \neq 0$, but $H^0(X, E \otimes L) = 0$ for any line bundle L on X with degree$(L) < 0$.

In [Har2, Chapter V, Section 2], the invariant e of $\mathrm{Gr}(E)$ is defined as $e := -\deg(E)$, after normalizing E as above. Very precise results on Seshadri constants are known when $e > 0$ (see [Ga, Sy]); Theorem 3.3 recovers these results.

We first claim that E is not semistable if and only if $e > 0$ which in turn is equivalent to degree$(E) < 0$, by definition of e.

Indeed, if $H^0(X, E) \neq 0$, then $\mathcal{O}_X$ is a subsheaf of E of slope zero, so if degree$(E) < 0$, then E can not be semistable. On the other hand, if E is not semistable, consider the Harder-Narasimhan filtration (2.1) of E:

$$0 = E_0 \subset E_1 \subset E_2 = E,$$

where E_1 is a sub-line-bundle of E with maximal degree. Then degree$(E_1) = 0$. For, since E has nonzero sections, degree$(E_1) \geq 0$. But as E has no nonzero sections after tensoring with any negative degree line bundle, degree of E_1 has to be zero. Now it follows that degree$(E) < 0$, since E_1 destabilizes E.

So this is a special case of our set-up with $r = 1 = \mathrm{rank}(E/E_1)$. By (2.3), we have $\theta = \deg(E/E_1) = \deg(E) < 0$. Further note that the hypothesis in Theorem 3.3 is

satisfied. In fact, we have an equality $\mu(E_2/E_1) - \mu(E_1/E_0) = \theta$. So Theorem 3.3 applies and we obtain precise values of Seshadri constants for any ample line bundle on $\mathrm{Gr}(E)$.

We now give four examples which satisfy the hypotheses of either Theorem 3.1 or Theorem 3.3. As far as we know, in all four examples our results give new computations of Seshadri constants. In our examples, $\mathrm{rank}(E) = 4$.

Example 4.2. Let X be a smooth complex projective curve. Let L be a line bundle of degree -1 on X, and set $E = L \oplus \mathcal{O}_X^{\oplus 3}$. Then $\mathrm{degree}(E) = -1$ and $\mu(E) = -\frac{1}{4}$. Since E has a subbundle of degree zero, it is not semistable. The Harder-Narasimhan filtration of E is

$$0 = E_0 \subset E_1 = \mathcal{O}_X^{\oplus 3} \subset E_2 = E.$$

Let $r := \mathrm{rank}(E_2/E_1) = 1$. Note that $\theta = \mathrm{degree}(L) = -1$, and E is normalized as in (2.18).

Since $\mu(E_2/E_1) - \mu(E_1/E_0) = -1 - 0 = \theta$, the hypotheses of Theorem 3.3 are satisfied. Thus this theorem precisely computes the Seshadri constants of line bundles on $\mathrm{Gr}(E)$.

Note that the sections of $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ are given by global sections of E, and there are three linearly independent global sections of E given by the three copies of $\mathcal{O}_X$ in E. These three sections of $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ meet precisely in Γ_s, which is the section of the projection $\mathrm{Gr}(E) \longrightarrow X$ given by the rank 1 quotient $E \longrightarrow L$. So Γ_s is the only curve in $\mathrm{Gr}(E)$ which meets $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ negatively. In this example, this can also be seen by observing that $E \longrightarrow L$ is the only rank 1 quotient of E which has negative degree.

Example 4.3. Let X be a smooth complex projective curve. Let L be a line bundle of degree -1 on X, and set $E = L^{\oplus 2} \oplus \mathcal{O}_X^{\oplus 2}$. Then $\mu(E) = -\frac{1}{2}$, and E is not semistable because it has a subbundle of degree zero. The Harder-Narasimhan filtration of E is

$$0 = E_0 \subset E_1 = \mathcal{O}_X^{\oplus 2} \subset E_2 = E.$$

Let $r := \mathrm{rank}(L^{\oplus 2}) = 2$. Note that $\theta = \mathrm{degree}(L^{\oplus 2}) = -2$, and E is normalized as in (2.18).

Since $\mu(E_2/E_1) - \mu(E_1/E_0) = -1 - 0 > \theta$, the hypotheses of Theorem 3.3 are not satisfied. Indeed, $\mathrm{Gr}(E)$ contains curves which meet $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ negatively, but are different from Γ_s. For example, the section of $\mathrm{Gr}(E) \longrightarrow X$ corresponding to the rank 2 quotient $E \longrightarrow \mathcal{O}_X \oplus L$ has intersection -1 with $\mathcal{O}_{\mathrm{Gr}(E)}(1)$.

On the other hand, the hypotheses of Theorem 3.1 are satisfied, because $\mathrm{rank}(E_1) = 2$ and $\mathrm{degree}(E_1) = 0$. Hence by Theorem 3.1, we can find precise values of Seshadri constants of ample line bundles on $\mathrm{Gr}(E)$ at points not contained in the base locus of $\mathcal{O}_{\mathrm{Gr}(E)}(1)$. In fact, we have precise values at points not contained in curves which meet $\mathcal{O}_{\mathrm{Gr}(E)}(1)$ negatively.

Example 4.4. Let X be a smooth complex projective curve. Let L_{-1} be a line bundle of degree -1 on X, and let L_1 be a line bundle of degree 1. Set $E = L_{-1}^{\oplus 2} \oplus L_1^{\oplus 2}$. Then $\mu(E) = 0$, and E is not semistable because it has a sub-line-bundle of degree 1. The Harder-Narasimhan filtration of E is

$$0 = E_0 \subset E_1 = L_1^{\oplus 2} \subset E_2 = E.$$

Let $r := \operatorname{rank}(E_2/E_1) = \operatorname{rank}(L_{-1}^{\oplus 2}) = 2$. Note that

$$\theta = \operatorname{degree}(E_2/E_1) = \operatorname{degree}(L_{-1}^{\oplus 2}) = -2,$$

and E is normalized as in (2.18).

Since $\mu(E_2/E_1) - \mu(E_1/E_0) = -1 - 1 = \theta$, the hypotheses of Theorem 3.3 are satisfied. Thus we can precisely compute the Seshadri constants of line bundles on $\operatorname{Gr}(E)$.

Example 4.5. Let X be a smooth complex projective curve of positive genus. Let L_{-1} be a line bundle of degree -1 on X, and let L_0 be a non-trivial line bundle of degree 0. Set $E = L_{-1} \oplus \mathcal{O}_X^{\oplus 2} \oplus L_0$. Then $\mu(E) = -\frac{1}{4}$ and E is not semistable because it has subbundles of degree 0. The Harder-Narasimhan filtration of E is

$$0 = E_0 \subset E_1 = \mathcal{O}_X^{\oplus 2} \oplus L_0 \subset E_2 = E.$$

Let $r := \operatorname{rank}(E_2/E_1) = \operatorname{rank}(L_{-1}) = 1$. Note that $\theta = \operatorname{degree}(E_2/E_1) = -1$, and E is normalized as in (2.18).

Since $\mu(E_2/E_1) - \mu(E_1/E_0) = -1 - 0 = \theta$, the hypotheses of Theorem 3.3 are satisfied. Thus we can precisely compute the Seshadri constants of line bundles on $\operatorname{Gr}(E)$.

Unlike in Example 4.2, the line bundle $\mathcal{O}_{\operatorname{Gr}(E)}(1)$ has only two independent sections. So the base locus of $\mathcal{O}_{\operatorname{Gr}(E)}(1)$ strictly contains Γ_s, which is the section of $\operatorname{Gr}(E) \longrightarrow X$ corresponding to the quotient $E \longrightarrow L_{-1}$. However, note that Γ_s is the only curve in $\operatorname{Gr}(E)$ which meets $\mathcal{O}_{\operatorname{Gr}(E)}(1)$ negatively, by Corollary 2.6.

Acknowledgements

References

[Ba]   T. Bauer, Seshadri constants and periods of polarized abelian varieties, *Math. Ann.* **312** (1998), 607–623.

[BDHKKSS] T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek and T. Szemberg, A primer on Seshadri constants, *Interactions of classical and numerical algebraic geometry*, 33–70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[BSS]   M. Beltrametti, S. Schneider and A. Sommese, Applications of the Ein-Lazarsfeld criterion for spannedness of adjoint bundles, *Math. Z.* **214** (1993), 593–599.

[BHP]   I. Biswas, A. Hogadi and A. J. Parameswaran, Pseudo-effective cone of Grassmann bundles over a curve, *Geom. Dedicata* **172** (2014), 69–77.

[BP]   I. Biswas and A. J. Parameswaran, Nef cone of flag bundles over a curve, *Kyoto Jour. Math.* **54** (2014), 353–366.

[BS]   I. Biswas and S. Subramanian, Semistability and finite maps, *Arch. Math.* **93** (2009), 437–443.

[CN]   P. Cascini and M. Nakamaye, Seshadri constants on smooth threefolds, *Adv. Geom.* **14** (2014), 59–79.

[Deb]   O. Debarre, Seshadri constants of abelian varieties, *The Fano Conference*, 379–394, Univ. Torino, Turin, 2004.

[Dem]   J.-P. Demailly, Singular Hermitian metrics on positive line bundles, *Complex algebraic varieties* (Bayreuth, 1990), 87–104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[EKL]   L. Ein, O. Küchle and R. Lazarsfeld, Local positivity of ample line bundles, *J. Differential Geom.* **42** (1995), 193–219.

[EL]   L. Ein and R. Lazarsfeld, Seshadri constants on smooth surfaces, *Journées de Géométrie Algébrique d'Orsay* (Orsay, 1992), *Astérisque* No. 218 (1993), 177–186.

[Ga]   L. F. García, Seshadri constants on ruled surfaces: the rational and the elliptic cases, *Manuscr. Math.* **119** (2006), 483–505.

[Hac]   C. Hacon, Remarks on Seshadri constants of vector bundles, *Ann. Inst. Fourier* **50** (2000), 767–780.

[HM]   K. Hanumanthu and A. Mukhopadhyay, Multi-point Seshadri constants on ruled surfaces, *Proc. Amer. Math. Soc.* **145** (2017), 5145–5155.

[Har1] R. Hartshorne, Robin, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-New York 1970.

[Har2] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[It1]   A. Ito, Okounkov bodies and Seshadri constants, *Adv. Math.* **241** (2013), 246–262.

[It2]   A. Ito, Seshadri constants via toric degenerations, *J. Reine Angew. Math.* **695** (2014), 151–174.

[La1]   R. Lazarsfeld, Lengths of periods and Seshadri constants of abelian varieties, *Math. Res. Lett.* **3** (1996), 439–447.

[La2]   R. Lazarsfeld, *Positivity in algebraic geometry I Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 48, Springer-Verlag, Berlin, 2004.

[Le]   S. Lee, Seshadri constants and Fano manifolds, *Math. Zeit.* **245** (2003), 645–656.

[Na]   M. Nakamaye, Seshadri constants on abelian varieties, *Amer. J. Math.* **118** (1996), 621–635.

[Og]   K. Oguiso, *Seshadri constants in a family of surfaces*, *Math. Ann.* **323** (2002), 625–631.

[Sy]   W. Syzdek, *Seshadri constants and geometry of surfaces*, Ph.D thesis, 2005, University of Duisburg-Essen.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in

Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India

E-mail address: krishna@cmi.ac.in

The Institute of Mathematical Sciences (HBNI), CIT Campus, Taramani, Chennai 600113, India; Indian Institute of Science Education and Research, Tirupati, Rami Reddy Nagar, Karakambadi Road, Mangalam (P.O.) Tirupati 517507, India

E-mail address: dsn@imsc.res.in and dsn@iisertirupati.ac.in

Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool, L69 7ZL, England

E-mail address: newstead@liverpool.ac.uk