SESHADRI CONSTANTS OF CURVE CONFIGURATIONS ON SURFACES

KRISHNA HANUMANTHU, PRAVEEN KUMAR ROY, AND ADITYA SUBRAMANIAM

Abstract. Let X be a complex nonsingular projective surface and let L be an ample line bundle on X. We study multi-point Seshadri constants of L at singular points of certain arrangements of curves on X. We pose some questions about such Seshadri constants and prove some results in the case of star arrangements of curves. We also study the configurational Seshadri constants for curve arrangements on surfaces and compare them with the usual Seshadri constants. We give several examples illustrating the properties that we study.

1. Introduction

In this note, we study multi-point Seshadri constants of ample line bundles centered at the singular loci of certain curve arrangements on surfaces. We recall the notion of multi-point Seshadri constants briefly below.

Let X be a smooth complex projective variety and let L be a nef line bundle on X. Let $r \geq 1$ be an integer and let $x_1, \ldots, x_r$ be distinct points of X. The *Seshadri constant* of L at $x_1, \ldots, x_r \in X$ is defined as:

$$\varepsilon(X, L, x_1, \ldots, x_r) := \inf_{\substack{C \subset X \text{ a curve with} \\ C \cap \{x_1, \ldots, x_r\} \neq \emptyset}} \frac{L \cdot C}{\sum_{i=1}^{r} \mathrm{mult}_{x_i} C}.$$

It is easy to see that the infimum above is the same as the infimum taken over irreducible, reduced curves C such that $C \cap \{x_1, \ldots, x_r\} \neq \emptyset$.

The following is a well-known upper bound for Seshadri constants. Let n be the dimension of X. Then for any $x_1, \ldots, x_r \in X$,

$$\varepsilon(X, L, x_1, \ldots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}.$$

The study of multi-point Seshadri constants is interesting even in the case of the projective plane $\mathbb{P}^2$. A famous conjecture of Nagata asserts that if $r \geq 10$ and $x_1, \cdots, x_r \in \mathbb{P}^2$ are very general, then

$$\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), x_1, \ldots, x_r) = \frac{1}{\sqrt{r}}.$$

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Nagata [11] proved this when $r = s^2$ for an integer s, but it is open in all other cases.

On the other hand, when the points $x_1, \cdots, x_r$ lie in special position, the corresponding Seshadri constant is frequently rational, and it is a very interesting problem to compute it. For example, if the points are collinear then $\varepsilon(X, L, x_1, \ldots, x_r) = \frac{1}{r}$.

In this direction, the author in [13] considers behaviour of multi-point Seshadri constants of $\mathcal{O}_{\mathbb{P}^2}(1)$ at the set of singular points of a line configuration in the plane $\mathbb{P}^2$. The following question was posed in [13].

Question 1.1. Let $x_1, \cdots, x_r \subset \mathbb{P}^2$ be the set of singular points in a configuration of lines which is not a pencil of lines. Let k denote the maximum number of points which lie on a line. Is it true that

$$\varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), x_1, \ldots, x_r) = \frac{1}{k}?$$

In [13], the author gives specific examples like line arrangements satisfying *Hirzebruch's property* and star arrangement of d lines in $\mathbb{P}^2$ where there is an affirmative answer to the above question. In [9], the authors extend the above study to the case of arrangements of plane curves of a fixed degree. These arrangements were introduced in the context of Harbourne constants in [14] and the bounded negativity conjecture. In order to study the local negativity phenomenon for algebraic surfaces, it was more reasonable to consider curve arrangements instead of irreducible curves as they are more difficult to construct.

In [9], the authors also study the multi-point Seshadri constants of $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ centered at singular loci of certain curve arrangements $\mathcal{C}$ in $\mathbb{P}^2_{\mathbb{C}}$ using a combinatorial invariant called the *configurational Seshadri constant of* $\mathcal{C}$. They give lower bounds for the configurational Seshadri constants of $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$ and also provide some actual values of the multi-point Seshadri constants for some classes of curve arrangements, comparing them with their associated configurational Seshadri constants.

In this paper, we continue this study by looking at curve arrangements on arbitrary surfaces. We prove some analogues of the results in [9] for arbitrary surfaces. We work over the field of complex numbers.

We recall below the basic definitions that we require in this paper.

Definition 1.2 (Transversal arrangement)**.** Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be an arrangement of curves on a smooth projective surface X. We say that $\mathcal{C}$ is a *transversal arrangement* if $d \geq 2$, all curves C_i are smooth, irreducible and they intersect pairwise transversally.

Let $\mathrm{Sing}(\mathcal{C})$ be the set of all intersection points of the curves in a transversal arrangement $\mathcal{C}$. Let s denote the number of points in $\mathrm{Sing}(\mathcal{C})$.

Definition 1.3 (Combinatorial invariants of transversal arrangements)**.** Let $\mathcal{C}$ be a transversal arrangement on a smooth surface X. For a point $p \in X$, let r_p denote the number of elements of $\mathcal{C}$ that pass through p. We call r_p the *multiplicity* of p in $\mathcal{C}$. We say p is a k-fold point of $\mathcal{C}$ if there are exactly k curves in $\mathcal{C}$ passing through p. For a positive integer $k \geq 2$, t_k denotes the number of k-fold points in $\mathcal{C}$.

These numbers satisfy the following standard equality, which follows by counting incidences in a transversal arrangement in two ways:

$$\sum_{i<j}(C_i \cdot C_j) = \sum_{k\geq 2}\binom{k}{2}t_k. \tag{1.1}$$

Also, let

$$f_i = f_i(\mathcal{C}) := \sum_{k\geq 2} k^i t_k.$$

In particular, $f_0 = s$ is the number of points in $\mathrm{Sing}(\mathcal{C})$.

Definition 1.4. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be an arrangement of irreducible curves on a smooth projective surface X and singular locus $\mathrm{Sing}(\mathcal{C})$. Let L be an ample line bundle on X. We define the *configurational Seshadri constant* of $\mathcal{C}$ as

$$\varepsilon_{\mathcal{C}}(L) := \frac{\sum\limits_{i=1}^{d} L \cdot C_i}{\sum\limits_{p\in \mathrm{Sing}(\mathcal{C})} \mathrm{mult}_p(\mathcal{C})}.$$

The main configurations considered in this paper are the *star configurations*. We make the following definition:

Definition 1.5. Let X be a nonsingular projective surface. A transversal arrangement of curves $\mathcal{C}$ on X is called a *star configuration* if the only intersection points are double points.

If $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ is a star arrangement and C_i are linearly equivalent to each other then

$$t_2 = \frac{C_1^2 d(d-1)}{2} \text{ and } t_k = 0 \text{ for } k > 2.$$

Star configurations of lines in $\mathbb{P}^2$, and more generally of hyperplanes in $\mathbb{P}^n$, are extensively studied; see [2, 4, 9], for some results.

In Section 2, we prove our main result (Theorem 2.5) for star arrangements of curves on arbitrary surfaces. In Section 2.1, we consider some examples on ruled surfaces. In Section 2.2, we compare the usual Seshadri constants with configurational Seshadri constants. In Section 2.3, we give lower bounds for configurational Seshadri constants for curve arrangements on surfaces. Finally, we give an example of a computation of multi-point Seshadri constant at the singular locus of an arrangement of curves on a K3 surface.

2. Seshadri constants for certain transversal curve arrangements

We work with curve configurations on surfaces satisfying the following assumption.

Assumption 2.1. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$, $d \geq 4$ be a transversal arrangement of curves on a complex smooth projective surface X with the following properties:

- There is no point where all the curves meet, i.e., $t_d = 0$.
- All the curves C_i in $\mathcal{C}$ are linearly equivalent to a fixed divisor A on X.

Definition 2.2. Let $\mathcal{C}$ be a transversal arrangement of curves on a smooth projective surface X satisfying Assumption 2.1. Then we define the *base constant* of $\mathcal{C}$ as

$$\mathrm{bs}(\mathcal{C}) := \max\{t \,|\, t = \#\, C_i \cap \mathrm{Sing}(\mathcal{C}),\, C_i \in \mathcal{C}\},$$

i.e., $\mathrm{bs}(\mathcal{C})$ is equal to the maximum number of singular points of $\mathcal{C}$ that are contained in a single curve $C_i \in \mathcal{C}$.

The following question is a generalization of Question 1.1 to curve arrangements on arbitrary surfaces.

Question 2.3. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of curves on a smooth projective surface X satisfying Assumption 2.1. Let L be an ample line bundle on X. Is it true that

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) = \frac{L \cdot C_1}{\mathrm{bs}(\mathcal{C})}?$$

By [9, Example 3.5], the above question has a negative answer for the Hesse arrangement of conics in $\mathbb{P}^2$. However it is open for line arrangements in $\mathbb{P}^2$. It is interesting to study the situations in which it is true on arbitrary surfaces.

For an arbitrary curve arrangement $\mathcal{C}$ satisfying Assumption 2.1, the base constant is at most $C_1^2(d-1)$. So the following question arises naturally.

Question 2.4. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_d\}$ be a transversal arrangement of curves on a smooth projective surface X satisfying Assumption 2.1. Let $\mathrm{Sing}(\mathcal{C})$ denote the singular locus of $\mathcal{C}$ and let L be an ample line bundle on X. Is it true that

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) \geq \frac{L \cdot C_1}{C_1^2(d-1)}? \tag{2.1}$$

For star arrangements, the base constant is exactly equal to $C_1^2(d-1)$. We now show that the lower bound in Question 2.4 can be achieved by certain star configuration of curves on X. So Question 2.3 has an affirmative answer for these star arrangements.

Theorem 2.5. *Let X be a nonsingular projective surface and let $\mathcal{C} = \{C_1, \ldots, C_d\}$ be a star configuration of curves on X with $d \geq 4$ such that each C_i is linearly equivalent to a fixed divisor A on X. Let L be an ample line bundle on X such that $\frac{dC_1^2}{(d-1)(L\cdot C_1)}L - C_1$ is nef. Then*

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) = \frac{L \cdot C_1}{C_1^2(d-1)}.$$

Proof. Let $\mathcal{P} = \{p_1, \ldots, p_s\}$ be the singular locus of $\mathcal{C}$. Note that $p_1, \ldots, p_s$ are all double points of $\mathcal{C}$. Exactly $C_1^2(d-1)$ of these points lie on C_1. So the Seshadri ratio computed by C_1 is precisely $\frac{L\cdot C_1}{C_1^2(d-1)}$. Note that the same ratio is also contributed by each of the curves $C_2, \ldots, C_d$.

Suppose that there exists an irreducible and reduced curve D, different from C_i for $i \in \{1, \ldots, d\}$, having multiplicity $m_p(D)$ at each point $p \in \mathcal{P}$ such that

$$\frac{L \cdot D}{\sum_{p\in\mathcal{P}} m_p(D)} < \frac{L \cdot C_1}{C_1^2(d-1)}.$$

This gives

$$(\triangle): \quad (L\cdot D)C_1^2(d-1) < \left(\sum_{p\in\mathcal{P}} m_p(D)\right) L\cdot C_1.$$

By Bézout's theorem, we obtain

$$\begin{aligned} D\cdot(C_1+\ldots+C_d) = d(D\cdot C_1) &\geq \sum_{p\in\mathcal{P}} m_p(D)\cdot m_p(C_1+\ldots+C_d)\\ &\overset{(*)}{\geq} 2\sum_{p\in\mathcal{P}} m_p(D)\\ &\overset{(\triangle)}{>} \frac{2(L\cdot D)C_1^2(d-1)}{L\cdot C_1}, \end{aligned}$$

where $(*)$ comes from the fact that all the singular points of $C_1+\ldots+C_d$ are at least double points. Now we use the nefness of $\frac{dC_1^2}{(d-1)(L\cdot C_1)}L - C_1$ and note the following inequality:

$$\frac{dC_1^2}{(d-1)(L\cdot C_1)} L\cdot D \geq D\cdot C_1.$$

This leads to the following:

$$\begin{aligned} \frac{d^2C_1^2}{(d-1)(L\cdot C_1)} L\cdot D &\geq d(D\cdot C_1) > \frac{2(d-1)(L\cdot D)C_1^2}{L\cdot C_1}\\ \Rightarrow \frac{d^2C_1^2}{d-1} &> 2(d-1)C_1^2 \qquad \text{(since } L \text{ is ample)}\\ \Rightarrow \left(\frac{d}{d-1}\right)^2 &> 2 \qquad \text{(since } C_1^2 = C_1\cdot C_2 > 0\text{)}. \end{aligned}$$

This is a contradiction, since $d \geq 4$. □

Corollary 2.6. *If $\mathcal{C} = \{C_1,\ldots,C_d\}$ is a star configuration on a nonsingular projective surface X with $d\geq 4$ and L is an ample line bundle on X such that $C_i \in |mL|$ for all i and a positive integer $m>0$, then*

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) = \frac{1}{m(d-1)}.$$

Remark 2.7. Several authors have studied multi-point Seshadri constants on surfaces; for a sample of results, see [1, 5, 7, 15, 16, 17]. Many of these results consider general points. Theorem 2.5 gives a computation of Seshadri constants at *special* points.

The proof of Theorem 2.5 shows that for an arbitrary arrangement, not necessarily a star arrangement, we have the following lower and upper bounds for the Seshadri constants.

Corollary 2.8. *Let X be a nonsingular projective surface and let $\mathcal{C} = \{C_1,\ldots,C_d\}$ be a configuration of curves on X satisfying Assumption 2.1. Let L be an ample line bundle on X such that $\frac{dC_1^2}{(d-1)(L\cdot C_1)}L - C_1$ is nef. Then*

$$\frac{L\cdot C_1}{C_1^2(d-1)} \leq \varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) \leq \frac{L\cdot C_1}{\mathrm{bs}(\mathcal{C})}.$$

Proof. The inequality $\frac{L\cdot C_1}{C_1^2(d-1)} \leq \varepsilon(X, L, \mathrm{Sing}(\mathcal{C}))$ follows from the proof of Theorem 2.5. Note that for any reduced and irreducible curve D which is not in $\mathcal{C}$, we showed that the Seshadri ratio is at least $\frac{L\cdot C_1}{C_1^2(d-1)}$ and this part does not use the fact that $\mathcal{C}$ is a star arrangement.

For curves in $\mathcal{C}$, the least Seshadri ratio is $\frac{L\cdot C_1}{\mathrm{bs}(\mathcal{C})}$. Since we always have $\mathrm{bs}(\mathcal{C}) \leq C_1^2(d-1)$, the proof is complete. □

Example 2.9. Consider the Hesse arrangement $\mathcal{C} = \{C_1, \ldots, C_{12}\}$ of conics in $\mathbb{P}^2$ which has 21 singular points. The base constant of $\mathcal{C}$ is 8. The Seshadri constant of $L = \mathcal{O}_{\mathbb{P}^2}(1)$ at these 21 points is known to be $\frac{1}{5}$; see [9, Example 3.5].

It is easy to see that $\frac{24}{11}L - C_1$ is nef. Then Corollary 2.8 implies

$$\frac{1}{22} < \varepsilon(\mathbb{P}^2, L, \mathrm{Sing}(\mathcal{C})) = \frac{1}{5} < \frac{1}{4}.$$

This shows that the inequalities in Corollary 2.8 are strict, in general. In particular, Theorem 2.5 does not hold in this case. Note that $\mathcal{C}$ is not a star arrangement.

Example 2.10. Let $\mathcal{L}_n \subset \mathbb{P}^2$ denote the n-th Fermat arrangement of $3n$ lines l_i with $n \geq 3$ given by the linear factors of the polynomial

$$Q(x, y, z) = (x^n - y^n)(y^n - z^n)(z^n - x^n).$$

It is well known that it has n^2 (say $\{p_1, \ldots, p_{n^2}\}$) triple points and 3 points of multiplicity n (say $\{q_1, q_2, q_3\}$); see [18, Example II.6]. We now consider a curve C of degree 6 not passing through either p_i or q_j and take the double cover $\pi : X \longrightarrow \mathbb{P}^2$ branched along C. Then X is a K3 surface and $L := \pi^*(\mathcal{O}(1))$ is ample.

For each i, let C_i denote the inverse image of l_i and let $\mathcal{C}_n := \{C_1, \ldots, C_{3n}\}$ be an arrangement of curves on X. Note that $C_i^2 = 2$ for every i and C_i is linearly equivalent to C_j for all i, j. One can check that the base constant of the arrangement $\mathcal{C}_n$ is $2n + 2$.

It is easy to see that this arrangement satisfies the hypotheses of Corollary 2.8. So we have the following inequalities:

$$\frac{1}{3n-1} \leq \varepsilon(X, L, \mathrm{Sing}(\mathcal{C}_n)) \leq \frac{1}{n+1}.$$

We now construct examples such that the conclusion of Theorem 2.5 holds, even without the nefness condition in the hypothesis. We construct these examples by obtaining transversal arrangement of curves on rational ruled surfaces coming from line arrangements in $\mathbb{P}^2$.

2.1. **Examples on ruled surfaces.**

We first recall some basic facts about ruled surfaces. We follow the notation of [8, Chapter V, Section 2].

Let C be a smooth complex curve of genus g and let $\pi : X \to C$ be a ruled surface over C. We choose a normalized vector bundle E of rank 2 on C such that $X = \mathbb{P}_C(E)$. Let $e := -\deg(E)$. Let C_0 be the image of a section of π such that $C_0^2 = -e$ and let f be a fiber of π. Then the Picard group of X modulo numerical equivalence is a free abelian group of rank 2 generated by C_0 and f. We have $C_0 \cdot f = 1$ and $f^2 = 0$.

A complete characterization of ample line bundles and irreducible curves on X is known. For this, and other details on ruled surfaces, we refer to [8, Chapter V, Section 2].

Let $X = X_e \to \mathbb{P}^1$ be a rational ruled surface with invariant $e \geq 1$. Given a line arrangement in $\mathbb{P}^2$, one can obtain an arrangement of curves on X_e, following a construction outlined in [3, Example 15], where a specific finite morphism $X_e \to X_1$ of degree e is described. Note that X_1 is isomorphic to a blow up of $\mathbb{P}^2$ at a point. So we can pull-back lines in $\mathbb{P}^2$ to X_e which are in the class $(1, e)$. If $\mathcal{L}$ is a line arrangement of d lines in the plane, its pull-back gives a curve arrangement $\mathcal{C}$ of d curves in X_e.

To be more precise, suppose that $\mathcal{L}$ has s singularities and t_k denotes the number of singular points of $\mathcal{L}$ of multiplicity k. Then the singular points of $\mathcal{C}$ are precisely the pre-images of singularities of $\mathcal{L}$. So $\mathcal{C}$ has es singular points and the number of singular points of multiplicity k is et_k. Note that each curve in $\mathcal{C}$ is in the class $(1, e)$ and has self-intersection e.

In the next example, the conclusion of Theorem 2.5 holds, even without the nefness condition in the hypothesis.

Example 2.11. Consider a transversal arrangement $\mathcal{C} = \{C_1, \ldots, C_5\}$ of curves on a rational ruled surface $X = X_e$ with invariant $e = 2$ such that $C_i \in |C_0 + 2f|$ and such that the only intersection points are ordinary double points. This configuration can be obtained as pullback of a star configuration of lines from $\mathbb{P}^2$, using the above construction. Note that each C_i contains exactly $(5-1)e = 8$ double points from the arrangement.

Let $L = C_0 + 3f$ be an ample line bundle on X. Then we have

$$(5C_1^2/4L \cdot C_1))L - C_1 = \frac{1}{6}(-C_0 + 3f),$$

which is not nef as $(-C_0 + 3f) \cdot f = -1 < 0$. Hence the nefness hypothesis in Theorem 2.5 fails.

We now verify that

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) = \frac{L \cdot C_1}{4C_1^2} = \frac{3}{8}.$$

Let $\mathcal{P} = \{p_1, \ldots, p_{20}\}$ denote the set of all the singular points of $\mathcal{C}$. Suppose that there exists an irreducible and reduced curve D, different from each C_i for $i \in \{1, \ldots, 5\}$, having multiplicity $m_p(D)$ at each point $p \in \mathcal{P}$ such that

$$\frac{L \cdot D}{\sum_{p \in \mathcal{P}} m_p(D)} < \frac{3}{8}. \tag{2.2}$$

Using Bézout's theorem, we have

$$\begin{aligned} D \cdot (C_1 + \ldots + C_5) = 5(D \cdot C_1) &\geq \sum_{p \in \mathcal{P}} m_p(D) \cdot m_p(C_1 + \ldots + C_5) \\ &= 2 \sum_{p \in \mathcal{P}} m_p(D) \\ &> \frac{16(L \cdot D)}{3}. \end{aligned}$$

Since C_1 is linearly equivalent to $L-f$ and f is nef, we have

$$5(D\cdot L)\geq 5(D\cdot C_1)>\frac{16(L\cdot D)}{3},$$

which is absurd since L is ample.

Example 2.12. We now construct examples where the inequality in Equation 2.1 is a strict inequality. Consider a transversal arrangement $\mathcal{C}_n=\{C_1,\ldots,C_{3n}\}$ of curves on a rational ruled surface $X=X_e$ with invariant $e\geq n>1$ such that $C_i\in|C_0+ef|$ and such that the curves C_i come as a pullback of a line arrangement $\mathcal{L}\subset\mathbb{P}^2$ satisfying Hirzebruch's property and having singular points with multiplicity greater than two, using the above construction, for more details, see [13]. Recall that a line arrangement $\mathcal{L}\subset\mathbb{P}^2$ satisfies Hirzebruch's property if the number of lines is equal to $3n$ for some $n\in\mathbb{Z}_{>0}$ and each line from $\mathcal{L}$ intersects any other line at exactly $n+1$ points.

Let $L=C_0+(e+1)f$ be an ample line bundle on X. We now verify that

$$\varepsilon(X,L,\mathrm{Sing}(\mathcal{C}_n))=\frac{L\cdot C_1}{e(n+1)}=\frac{e+1}{e(n+1)}.$$

Note that $\frac{e+1}{e(n+1)}>\frac{e+1}{e(3n-1)}$, which shows that the inequality in Equation 2.1 is strict.

Let $\mathcal{P}$ denote the set of all intersection points of $\mathcal{C}_n$. The Seshadri ratio computed by C_1 is precisely $\frac{e+1}{e(n+1)}$. Suppose that there exists an irreducible and reduced curve D, different from each C_i for $i\in\{1,\ldots,3n\}$, having multiplicity $m_p(D)$ at each point $p\in\mathcal{P}$ such that

$$\frac{L\cdot D}{\sum_{p\in\mathcal{P}}m_p(D)}<\frac{e+1}{e(n+1)}. \tag{2.3}$$

Using Bézout's theorem, we have

$$\begin{aligned}D\cdot(C_1+\ldots+C_{3n})=3n(D\cdot C_1) &\geq \sum_{p\in\mathcal{P}}m_p(D)\cdot m_p(C_1+\ldots+C_{3n})\\ &\geq 3\sum_{p\in\mathcal{P}}m_p(D)\\ &> \frac{3(L\cdot D)e(n+1)}{e+1}.\end{aligned}$$

Since C_1 is linearly equivalent to $L-f$ and f is nef, we have

$$3n(D\cdot L)\geq 3n(D\cdot C_1)>\frac{3(L\cdot D)e(n+1)}{e+1},$$

which is not possible since $e\geq n$ by the choice of e.

Example 2.13. There are examples of line arrangements in $\mathbb{P}^2$ satisfying Hirzebruch's property and such that all the singular points have multiplicity greater than two; for more details, see [13].

First, we consider the n-th Fermat arrangement $\mathcal{L}_n$ of lines in $\mathbb{P}^2$ with $n\geq 3$. As noted above, this arrangement consists of $3n$ lines with $t_n=3$ and $t_3=n^2$. Note that $\mathcal{L}_n$ satisfies

Hirzebruch property. Using the above construction, we can pullback $\mathcal{L}_n$ to a curve arrangement $\mathcal{C}'_n$ on X_e with $e > n$. Then for the ample line bundle $L = C_0 + (e+1)f$ on X_e as in Example 2.12, we get

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{C}'_n)) = \frac{L \cdot C_1}{e(n+1)} = \frac{e+1}{e(n+1)}.$$

Next, let $\mathcal{K}$ denote the Klein arrangement of lines consisting of 21 lines in $\mathbb{P}^2$ with $t_4 = 21$ and $t_3 = 28$. Note that $\mathcal{K}$ satisfies Hirzebruch property. Using the above construction, we can pullback $\mathcal{K}$ to a curve arrangement $\mathcal{K}'$ on X_e with $e > n$ as in the above construction. Then for the ample line bundle L on X_e as in Example 2.12, we get

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{K}')) = \frac{L \cdot C_1}{e(n+1)} = \frac{e+1}{8e}.$$

We now consider another arrangement satisfying Hirzebruch property, namely the Wiman arrangement of lines $\mathcal{W}$ in $\mathbb{P}^2$ consisting of 45 lines with $t_3 = 120$, $t_4 = 45$ and $t_5 = 36$. Using the above construction, we can pullback $\mathcal{W}$ to a curve arrangement $\mathcal{W}'$ on X_e with $e > n$ as in the above construction. Then for the ample line bundle L on X_e as in Example 2.12, we get

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{W}')) = \frac{L \cdot C_1}{e(n+1)} = \frac{e+1}{16e}.$$

2.2. **Comparing the Seshadri constant and the configurational Seshadri constant.** We now study possible discrepancies between the configurational and the multi-point Seshadri constants with some examples. It is clear that for an arrangement $\mathcal{C}$ on a smooth projective surface X and an ample line bundle L, we always have the following inequality:

$$\varepsilon_{\mathcal{C}}(L) \geq \varepsilon(X, L, \mathrm{Sing}(\mathcal{C})).$$

In general, this inequality is strict.

Example 2.14. In Theorem 2.5, we considered a *star* configuration $\mathcal{C} = \{C_1, \ldots, C_d\}$ with $d \geq 4$ such that $C_i \in |A|$ for all i, where A was a fixed divisor on X. We saw that for an ample line bundle L on X, if $\frac{dC_1^2}{(d-1)(L\cdot C_1)}L - C_1$ is nef, the Seshadri constant is given by

$$\varepsilon(X, L, \mathrm{Sing}(\mathcal{C})) = \frac{L \cdot C_1}{C_1^2(d-1)}.$$

On the other hand, the configurational Seshadri constant is given by

$$\varepsilon_{\mathcal{C}}(L) = \frac{d(L \cdot C_1)}{2t_2} = \frac{L \cdot C_1}{C_1^2(d-1)}.$$

Thus in this case, both these constants agree.

Example 2.15. In Example 2.13, we first considered a curve arrangement $\mathcal{C}'_n$ on X_e with $e > n$ obtained as a pullback of the n-th Fermat arrangement of lines $\mathcal{L}_n \subset \mathbb{P}^2$ with $n \geq 3$. This arrangement $\mathcal{C}'_n$ consists of $3n$ curves with $t_n = 3e$ and $t_3 = en^2$. Then for the ample line bundle $L = C_0 + (e+1)f$ on X_e, we obtained the following value of Seshadri constant

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{C}'_n)) = \frac{L \cdot C_1}{e(n+1)} = \frac{e+1}{e(n+1)}.$$

On the other hand, the configurational Seshadri constant is given by

$$\varepsilon_{\mathcal{C}'_n}(L) = \frac{3n(L\cdot C_1)}{f_1} = \frac{e+1}{e(n+1)}.$$

Thus, in this case, both these constants agree.

Next, we considered a curve arrangement $\mathcal{K}'$ on X_e with $e > n$ which is a pullback of the Klein arrangement of lines $\mathcal{K} \subset \mathbb{P}^2$. The curve arrangement $\mathcal{K}'$ consists of 21 curves with $t_4 = 21e$ and $t_3 = 28e$. Then by the choice of an ample line bundle L on X_e as in Example 2.12, we obtained the following value of Seshadri constant

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{K}')) = \frac{e+1}{8e}.$$

On the other hand, the configurational Seshadri constant is given by

$$\varepsilon_{\mathcal{K}'}(L) = \frac{21(L\cdot C_1)}{f_1} = \frac{e+1}{8e}.$$

Thus, in this case too, both the constants agree.

We then considered the curve arrangement $\mathcal{W}'$ on X_e with $e > n$ obtained as a pullback of the Wiman arrangement of lines $\mathcal{W} \subset \mathbb{P}^2$. The curve arrangement $\mathcal{W}'$ consists of 45 curves with $t_3 = 120e$, $t_4 = 45e$ and $t_5 = 36e$. Then by the choice of an ample line bundle L on X_e as in Example 2.12, we get

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{W}')) = \frac{L\cdot C_1}{e(n+1)} = \frac{e+1}{16e}.$$

On the other hand, the configurational Seshadri constant is given by

$$\varepsilon_{\mathcal{W}'}(L) = \frac{45(L\cdot C_1)}{f_1} = \frac{e+1}{16e}.$$

Thus, in this case too, both the constants agree.

We give an example below where the Seshadri constant is strictly smaller than the configurational Seshadri constant.

Example 2.16. In [9, Example 3.1], it was shown that for a Hirzebruch quasi-pencil $\mathcal{H}$ of $k \geq 4$ lines in $\mathbb{P}^2_{\mathbb{C}}$ with $t_{k-1} = 1$ and $t_2 = k-1$, the multi-point Seshadri constant $\varepsilon(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1), \mathrm{Sing}(\mathcal{H}))$ is strictly smaller than its configurational value $\varepsilon_{\mathcal{H}}(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1))$. We can obtain a configuration $\mathcal{H}'$ on a rational ruled surface X_e with $e = k+1$, as a pullback of the Hirzebruch quasi-pencil $\mathcal{H}$, using the construction mentioned above. The configuration $\mathcal{H}'$ consists of $k \geq 4$ curves say $C_1, \ldots, C_k$, with $t_{k-1} = e$, $t_2 = (k-1)e$ and $C_i \in |C_0 + ef|$ for all i. Let $L = C_0 + (e+1)f$ be an ample line bundle on X_e.

In this case, the configurational Seshadri constant is

$$\varepsilon_{\mathcal{H}'}(L) = \frac{k(L\cdot C_1)}{2e(k-1)+(k-1)e} = \frac{k(e+1)}{3e(k-1)} = \frac{k(k+2)}{3(k^2-1)}.$$

We now claim that

$$\varepsilon(X_e, L, \mathrm{Sing}(\mathcal{H}')) = \frac{e+1}{e(k-1)} = \frac{k+2}{k^2-1}.$$

Indeed, we can assume that the points of multiplicity $k-1$ are defined by intersections of $C_1, \cdots, C_{k-1}$. Then the curve C_k gives the Seshadri ratio $\frac{e+1}{e(k-1)} = \frac{k+2}{k^2-1}$. Suppose that there exists an irreducible and reduced curve D, different from each C_i for each i, having multiplicity $m_p(D)$ at each point $p \in \mathcal{P} = \mathrm{Sing}(\mathcal{H}')$ such that

$$\frac{L \cdot D}{\sum_{p \in \mathcal{P}} m_p(D)} < \frac{e+1}{e(k-1)}.$$

Then

$$D \cdot (C_1 + C_k) = 2(D \cdot C_1) \geq \sum_{p \in \mathcal{P}} m_p(D) > \frac{e(k-1)(L \cdot D)}{e+1}.$$

Since C_1 is linearly equivalent to $L - f$, we have

$$2(D \cdot L) \geq 2(D \cdot C_1) > \frac{e(k-1)(L \cdot D)}{e+1},$$

i.e., $2(k+2) > k^2 - 1$. But this is not possible by the choice of k.

This shows that $\varepsilon_{\mathcal{H}'}(L) > \varepsilon(X_e, L, \mathrm{Sing}(\mathcal{H}'))$.

2.3. **Lower bounds on configurational Seshadri constants.** We now give some lower bounds on configurational Seshadri constants on ruled surfaces and on surfaces of non-negative Kodaira dimension.

Theorem 2.17. [6, Equation 4.9] *Let X be a ruled surface over a smooth curve of genus $g \geq 0$ with invariant $e \geq 4$. Let $\mathcal{C} = \{C_1, \ldots, C_d\}$ be a transversal arrangement of curves satisfying Assumption 2.1 such that each curve in $\mathcal{C}$ is numerically equivalent to $A = aC_0 + bf$ with $a > 0$ and $b \geq ae$. Assume further that either $a \geq 2$ or if $a = 1$, there exists a subset of four curves in $\mathcal{C}$ such that there is no point common to all the four curves. Then we have the following Hirzebruch-type inequality for $\mathcal{C}$:*

$$t_2 + \frac{3}{4}t_3 \geq -16 + 16g + \sum_{k \geq 5}(2k-9)t_k + d(e(5a^2 - 2a) - 10ab - 4ag + 4a + 4b).$$

Using the above result, we give a lower bound on configurational Seshadri constants.

Theorem 2.18. *Let X be a ruled surface over a smooth curve of genus $g \geq 0$ with invariant $e \geq 4$. Let $\mathcal{C} = \{C_1, \ldots, C_d\}$ be a transversal arrangement of curves satisfying hypothesis of Theorem 2.17 and let L an ample line bundle on X. Then*

$$\varepsilon_{\mathcal{C}}(L) \geq \frac{d(L \cdot C_1)}{8 - 8g + \frac{9(2ab - a^2e)d^2}{4} + \frac{d(2ae - \frac{a^2e}{2} + ab + 4ag - 4a - 4b)}{2}}.$$

Proof. Our strategy is based on the combinatorial features of $\mathcal{C}$. Let us denote $C = C_1 + \ldots + C_d$. Then we can write

$$\varepsilon_{\mathcal{C}}(L) = \frac{\sum_{i=1}^{d} L \cdot C_i}{\sum_{P \in \mathrm{Sing}(\mathcal{C})} \mathrm{mult}_P(\mathcal{C})} = \frac{d(L \cdot C_1)}{f_1}. \tag{2.4}$$

Our goal here is to find a reasonable upper-bound on the number $f_1 = \sum_{r \geq 2} r t_r$. In order to do so, we are going to use Theorem 2.17 and Hirzebruch's inequality, namely

$$t_2 + \frac{3}{4} t_3 \geq -16 + 16g + \sum_{k \geq 5} (2k-9) t_k + d(e(5a^2 - 2a) - 10ab - 4ag + 4a + 4b).$$

Simplifying this, we get

$$16 - 16g + d(2ae - 5a^2 e + 10ab + 4ag - 4a - 4b) + 9 f_0 \geq 2 f_1 + 4 t_2 + t_4 + \frac{9}{4} t_3.$$

Since $t_2, t_4, t_3 \geq 0$ we have

$$16 - 16g + d(2ae - 5a^2 e + 10ab + 4ag - 4a - 4b) + 9 f_0 \geq 2 f_1.$$

and hence

$$8 - 8g + \frac{d}{2}(2ae - 5a^2 e + 10ab + 4ag - 4a - 4b) + \frac{9}{2} f_0 \geq f_1.$$

Obviously one always has

$$d \leq f_0 \leq (2ab - a^2 e)\binom{d}{2}$$

which leads to

$$f_1 \leq 8 - 8g + \frac{9(2ab - a^2 e) d^2}{4} + \frac{d(2ae - \frac{a^2 e}{2} + ab + 4ag - 4a - 4b)}{2},$$

so finally we get

$$\varepsilon_{\mathcal{C}}(L) = \frac{d(L \cdot C_1)}{f_1} \geq \frac{d(L \cdot C_1)}{8 - 8g + \frac{9(2ab - a^2 e) d^2}{4} + \frac{d(2ae - \frac{a^2 e}{2} + ab + 4ag - 4a - 4b)}{2}}.$$

$\square$

We will now consider transversal configurations on surfaces of non-negative Kodaira dimension. We will use the following theorem.

Theorem 2.19. [10, Theorem 2.1] *Let X be a smooth complex projective surface with non-negative Kodaira dimension and let $C = C_1 + \cdots + C_d$ be a transversal configuration of smooth curves having $d \geq 2$ irreducible components $C_1, \cdots, C_d$. Then*

$$K_X \cdot C + 4 \sum_{i=1}^{d} (1 - g(C_i)) - t_2 + \sum_{r \geq 3} (r-4) t_r \leq 3 c_2(X) - K_X^2.$$

We now give a lower bound on configurational Seshadri constants on surfaces of non-negative Kodaira dimension.

Theorem 2.20. *Let X be a smooth complex projective surface with non-negative Kodaira dimension and let $C = C_1 + \cdots + C_d$ be a transversal configuration of smooth curves satisfying Assumption 2.1. Let L be an ample line bundle on X. Then*

$$\varepsilon_{\mathcal{C}}(L) \geq \frac{d(L \cdot C_1)}{3c_2(X) - K_X^2 + 4C_1^2\binom{d}{2} - K_X \cdot C - 4\sum_{i=1}^{d}(1 - g(C_i))}.$$

Proof. As before, we find an upper-bound on the number $f_1 = \sum_{r \geq 2} r t_r$. In order to do so, we are going to use Theorem 2.19 and Hirzebruch's inequality, namely

$$K_X \cdot C + 4\sum_{i=1}^{d}(1 - g(C_i)) - t_2 + \sum_{r \geq 3}(r-4)t_r \leq 3c_2(X) - K_X^2.$$

Simplifying, we get

$$K_X \cdot C + 4\sum_{i=1}^{d}(1 - g(C_i)) + t_2 + f_1 - 4f_0 \leq 3c_2(X) - K_X^2.$$

Since $t_2 \geq 0$, we have

$$K_X \cdot C + 4\sum_{i=1}^{d}(1 - g(C_i)) + f_1 - 4f_0 \leq 3c_2(X) - K_X^2$$

and hence

$$f_1 \leq 3c_2(X) - K_X^2 + 4f_0 - K_X \cdot C - 4\sum_{i=1}^{d}(1 - g(C_i)).$$

We know that $d \leq f_0 \leq C_1^2\binom{d}{2}$. So

$$f_1 \leq 3c_2(X) - K_X^2 + 4C_1^2\binom{d}{2} - K_X \cdot C - 4\sum_{i=1}^{d}(1 - g(C_i)).$$

Finally we obtain

$$\varepsilon_{\mathcal{C}}(L) = \frac{d(L \cdot C_1)}{f_1} \geq \frac{d(L \cdot C_1)}{3c_2(X) - K_X^2 + 4C_1^2\binom{d}{2} - K_X \cdot C - 4\sum_{i=1}^{d}(1 - g(C_i))}.$$

□

We end with an example of a configuration of lines on a K3 surface in $\mathbb{P}^3$. Though it does not fit in the situation studied in this paper (since the curves in the arrangement are not linearly equivalent to each other), it gives an interesting computation of a multi-point Seshadri constant on a K3 surface.

Example 2.21. Let $X \subset \mathbb{P}^3$ be a Fermat quartic defined by the vanishing locus of $x_0^4 + x_1^4 + x_2^4 + x_3^4$, where $x_0, \ldots, x_3$ are the coordinates of $\mathbb{P}^3$. Then X contains the following 48 lines; 16 from each of the following three types:

$$\begin{aligned} A &: \quad x_0 = \alpha x_1 \text{ and } x_2 = \beta x_3, \\ A' &: \quad x_0 = \alpha x_2 \text{ and } x_1 = \beta x_3, \\ A'' &: \quad x_0 = \alpha x_3 \text{ and } x_1 = \beta x_2, \end{aligned}$$

where $\alpha, \beta \in \{\zeta, -\zeta, i\zeta, -i\zeta\}$ and ζ is a primitive eighth root of unity.

We describe the intersection behaviour of these lines below.

First, we consider intersections among lines of type A. If

$$A_1: \ x_0 = \alpha x_1, \ x_2 = \beta x_3 \text{ and } B_1: \ x_0 = \alpha' x_1, \ x_2 = \beta' x_3$$

then

$$A_1 \cap B_1 \neq \phi \Leftrightarrow x_1 = 0 \text{ or } \alpha = \alpha' \text{ and } x_3 = 0 \text{ or } \beta = \beta'.$$

This shows that a line A_1 of type A meets three lines of the form $x_0 = \alpha x_1, \ x_2 = \beta' x_3$ $(\beta' \in \{\zeta, -\zeta, i\zeta, -i\zeta\} \backslash \{\beta\})$ of type A at $[\alpha : 1 : 0 : 0]$ and 3 other lines of type A of the form $x_0 = \alpha' x_1, \ x_2 = \beta x_3$ $(\alpha' \in \{\zeta, -\zeta, i\zeta, -i\zeta\} \backslash \{\alpha\})$ at $[0 : 0 : \beta : 1]$.

On the other hand, a line A_1 of type A meets only 4 out of the 16 lines of type A':

$$A_1 \cap \{x_0 = \alpha' x_2, x_1 = \beta' x_3\} \neq \emptyset$$

implies $x_0 = \alpha' x_2 = \alpha' \beta x_3$ and $x_0 = \alpha x_1 = \alpha \beta' x_3$. This shows $\alpha' \beta = \alpha \beta'$. Thus the choice of β' determines α', giving 4 lines of type A' . Moreover, the intersection point of A_1 and a line $\{x_0 = \alpha' x_2, x_1 = \beta' x_3\}$ of type A' is given by $[\alpha\beta' : \beta' : \beta : 1]$.

By a similar argument one can see that A_1 meets exactly 4 lines of type A''.

Now let $\mathcal{L} = \{l_1, l_2, \ldots, l_{48}\}$ be the arrangement consisting of all the 48 lines of type A, A' and A'' on X.

We claim that there exist 24 sets of lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\ldots$, $\mathcal{L}_{24}$ containing 4 lines each from $\mathcal{L}$ satisfying the following conditions. Let $1 \leq i \leq 24$.

- Each set $\mathcal{L}_i$ contains 4 lines of the same type.
- All the four lines in each set $\mathcal{L}_i$ meet in one point.
- The divisor obtained by adding the four lines in each $\mathcal{L}_i$ is linearly equivalent to $\mathcal{O}_X(1)$.
- Each l_i is contained in exactly two of the sets $\mathcal{L}_1$, $\mathcal{L}_2, \ldots, \mathcal{L}_{24}$.

We will prove that there are 8 sets of the above kind consisting of lines of type A. By a similar argument, we get the statement for types A' and A''. Fix two enumerations of the set $\{\zeta, -\zeta, i\zeta, -i\zeta\}$: $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$.

First, consider the line

$$l_{\alpha_1,\beta_1} = \{x_0 = \alpha_1 x_1 \text{ and } x_2 = \beta_1 x_3\}$$

of type A. From the intersection behaviour described above, we know that l_{α_1,β_1} meets exactly six lines of type A which are given by:

$$\begin{aligned} &l_{\alpha_1,\beta_i}, \quad \text{for} \quad i = 2, 3, 4, \text{ and} \\ &l_{\alpha_j,\beta_1}, \quad \text{for} \quad j = 2, 3, 4. \end{aligned}$$

The four lines $l_{\alpha_1,\beta_1}, l_{\alpha_1,\beta_2}, l_{\alpha_1,\beta_3}, l_{\alpha_1,\beta_4}$ meet at the point $[\alpha_1 : 1 : 0 : 0]$ and the four lines $l_{\alpha_1,\beta_1}, l_{\alpha_2,\beta_1}, l_{\alpha_3,\beta_1}, l_{\alpha_4,\beta_1}$ meet at the point $[0 : 0 : \beta_1 : 1]$.

We define

$$\begin{aligned} \mathcal{L}_{\alpha_1} &= \{l_{\alpha_1,\beta_1}, l_{\alpha_1,\beta_2}, l_{\alpha_1,\beta_3}, l_{\alpha_1,\beta_4}\}, \\ \mathcal{L}_{\beta_1} &= \{l_{\alpha_1,\beta_1}, l_{\alpha_2,\beta_1}, l_{\alpha_3,\beta_1}, l_{\alpha_4,\beta_1}\}. \end{aligned}$$

Similarly, we define $\mathcal{L}_{\alpha_i}$ and $\mathcal{L}_{\beta_j}$ for every $2 \leq i, j \leq 4$.

Note that every line of type A appears in exactly two of the eight sets $\mathcal{L}_{\alpha_i}$ and $\mathcal{L}_{\beta_j}$, where $1 \leq i, j \leq 4$. More precisely, a line l_{α_i,β_j} of type A is contained only in $\mathcal{L}_{\alpha_i}$ and in $\mathcal{L}_{\beta_j}$.

The collection of these $\mathcal{L}_{\alpha_i}$ and $\mathcal{L}_{\beta_j}$ contribute 8 sets of lines of type A. Similarly we can define $\mathcal{L}'_{\alpha_i}$ and $\mathcal{L}'_{\beta_j}$ from type A' and $\mathcal{L}''_{\alpha_i}$ and $\mathcal{L}''_{\beta_j}$ from type A''. This gives a collection of 24 sets which we rename as $\mathcal{L}_1$, $\mathcal{L}_2$, ..., $\mathcal{L}_{24}$ such that each l_i is exactly in two of this list.

We also note that all the lines in $\mathcal{L}_{\alpha_i}$ (respectively, in $\mathcal{L}_{\beta_j}$) lie in the hyperplane given by the vanishing of $x_0 - \alpha_i x_1$ (respectively, $x_2 - \beta_j x_3$) in $\mathbb{P}^3$. In fact, the intersection of these hyperplanes with X is precisely the four lines in question. Therefore, for $1 \leq i, j \leq 4$,

$$\begin{aligned} \mathcal{O}_X(1) &\sim l_{\alpha_i,\beta_1} + l_{\alpha_i,\beta_2} + l_{\alpha_i,\beta_3} + l_{\alpha_i,\beta_4}, \text{ and} \\ \mathcal{O}_X(1) &\sim l_{\alpha_1,\beta_j} + l_{\alpha_2,\beta_j} + l_{\alpha_3,\beta_j} + l_{\alpha_4,\beta_j}. \end{aligned}$$

This proves our claim.

From the intersection behaviour described above, it is clear that the only singular points for the arrangement $\mathcal{L}$ are either double points or quadruple points; [12, Example 3.3]. It is also not difficult to see that the number of singular points on any line in $\mathcal{L}$ is 10, as we show now. It suffices to verify this assertion for a line of type A. Given a line l_{α_i,β_j} of type A, we know that it has two quadruple points $[\alpha_i : 1 : 0 : 0]$ and $[0 : 0 : \beta_j : 1]$. We also know that each line of type A meets 4 lines each of type A' and A'' and these points are all double points. Therefore in all there are 10 singular points on a line of type A. By a similar argument one can show this for lines of type A' and type A'' also.

We now compute the multi-point Seshadri constant of $\mathcal{O}_X(1)$ at the singular locus of the arrangement $\mathcal{L}$. This arrangement has 216 singular points. So we have

$$\varepsilon(X, \mathcal{O}_X(1), \mathrm{Sing}(\mathcal{L})) \leq \sqrt{\frac{4}{216}} \sim 0.136.$$

In fact, we claim that

$$\varepsilon(X, \mathcal{O}_X(1), \mathrm{Sing}(\mathcal{L})) = \frac{1}{10}.$$

Note that each line l_{α_i,β_j} has degree 1 in $\mathbb{P}^3$. So, from the above discussion, it is clear that

$$\frac{\mathcal{O}_X(1) \cdot l_{\alpha_i,\beta_j}}{10} = \frac{1}{10}.$$

Now let $C \subset X$ be any reduced and irreducible curve such that $C \notin \mathcal{L}$ and which passes through some point in the singular locus of $\mathcal{L}$. By Bézout's theorem, we have the following inequality for every line $l \in \mathcal{L}$:

$$C \cdot l \geq \sum_{p \in l \cap \mathrm{Sing}(\mathcal{L})} \mathrm{mult}_p C.$$

We now sum such inequalities over all the 24 sets constructed above. Each such set contributes $C \cdot \mathcal{O}_X(1) = \deg(C)$ to the left hand side. On the right hand side, each line appears twice. Since each double point appears in two lines and each quadruple point appears in four lines, we obtain

$$\begin{aligned} 24\deg(C) &\geq 2\left(2 \sum_{p_i \in \mathrm{Sing}_2(\mathcal{L})} \mathrm{mult}_{p_i} C + 4 \sum_{q_j \in \mathrm{Sing}_4(\mathcal{L})} \mathrm{mult}_{q_j} C\right) \\ &\geq 4 \sum_{p \in \mathrm{Sing}(\mathcal{L})} \mathrm{mult}_p C. \end{aligned}$$

Here $\mathrm{Sing}_2(\mathcal{L})$ and $\mathrm{Sing}_4(\mathcal{L})$ denote the set of double points and the set of quadruple points of the arrangement $\mathcal{L}$, respectively. Therefore we get

$$\frac{\mathcal{O}_X(1) \cdot C}{\sum\limits_{p \in \mathrm{Sing}(\mathcal{L})} \mathrm{mult}_p C} = \frac{\deg(C)}{\sum\limits_{p \in \mathrm{Sing}(\mathcal{L})} \mathrm{mult}_p C} \geq \frac{1}{6} > \frac{1}{10}.$$

This completes the proof of the claim.

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References

[1] T. Bauer, Seshadri constants on algebraic surfaces, Math. Ann. **313** (1999), no.3, 547–583.
[2] E. Carlini, A. Van Tuyl, Star configuration points and generic plane curves, Proc. Amer. Math. Soc. **139** (2011), no.12, 4181–4192.
[3] S. Eterović, Logarithmic Chern slopes of arrangements of rational sections in Hirzebruch surfaces, Master Thesis, Pontificia Universidad Católica de Chile, Santiago 2015.
[4] A. V. Geramita, B. Harbourne, J. Migliore, Star configurations in $\mathbb{P}^n$, J. Algebra, **376** (2013), 279–299.
[5] K. Hanumanthu, A. Mukhopadhyay, *Multi-point Seshadri constants on ruled surfaces*, Proc. Amer. Math. Soc. **145** (2017), no. 12, 5145–5155.
[6] K. Hanumanthu, A. Subramaniam, Bounded negativity and Harbourne constants on ruled surfaces, Manuscripta Math. **164** (2021) 431–454.
[7] B. Harbourne, J. Roé, Discrete behavior of Seshadri constants on surfaces, J. Pure Appl. Algebra **212** (2008), no.3, 616–627.
[8] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977.
[9] M. Janasz and P. Pokora, On Seshadri constants and point-curve configurations, Electron. Res. Arch., **28(2)** (2020) 795–805.
[10] R. Laface and P. Pokora, Local negativity of surfaces with non-negative Kodaira dimension and transversal configurations of curves, Glasg. Math. J. **62** (2020), no.1, 123–135.
[11] M. Nagata, On the 14-th problem of Hilbert, Am. J. Math. **81** (1959), 766–772.
[12] P. Pokora, Harbourne constants and arrangements of lines on smooth hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$, Taiwanese J. Math., **20**, No. 1 (2016), 25–31.

[13] P. Pokora, Seshadri constants and special configurations of points in the projective plane, Rocky Mountain J. Math. **49(3)** (2019) 963 – 978.
[14] P. Pokora, X. Roulleau and T. Szemberg, Bounded negativity, Harbourne constants and transversal arrangements of curves, Ann. Inst. Fourier (Grenoble) **67** (2017), no. 6, 2719–2735.
[15] P. K. Roy, Some results on Seshadri constants on surfaces of general type, Eur. J. Math. **6** (2020), no.4, 1176–1190.
[16] J. Roé, J. Ross, An inequality between multipoint Seshadri constants, Geom. Dedicata **140** (2009), 175–181.
[17] W. Syzdek, T. Szemberg, Seshadri fibrations of algebraic surfaces, Math. Nachr. **283** (2010), no. 6, 902–908.
[18] G. A. Urzúa, Arrangements of curves and algebraic surfaces, Thesis (Ph.D.) University of Michigan, 2008, 166 pages.

Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India.

Email address: krishna@cmi.ac.in

UM-DAE Centre for Excellence in Basic Sciences "Nalanda", Opp Nano Sciences Building, University of Mumbai, Vidyanagari, Mumbai 400098, India.

Email address: praveen.roy@cbs.ac.in

Indian Institute of Science Education and Research Tirupati, Rami Reddy Nagar, Karakambadi Road, Mangalam (P.O.), Tirupati, Andhra Pradesh, INDIA – 517507.

Email address: adityasubramaniam@labs.iisertirupati.ac.in