# SESHADRI CONSTANTS ON BOTT TOWERS

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Abstract. For a positive integer $n$, let $X_n \to X_{n-1} \to \ldots \to X_2 \to X_1 \to X_0$ be a Bott tower of height $n$, and let $L$ be a nef line bundle on $X_n$. We compute Seshadri constants $\varepsilon(X_n, L, x)$ of $L$ at any point $x \in X_n$.

## 1. Introduction

Seshadri constants of line bundles reflect their local positivity. Soon after Demailly introduced Seshadri constants in [Dem], there has been extensive work on them; they have turned out to be important invariants. Let us briefly recall their definition.

Let $X$ be a complex projective variety, and let $L$ be a nef line bundle on $X$. For a point $x \in X$, the *Seshadri constant* of $L$ at $x$, denoted by $\varepsilon(X, L, x)$, is defined to be

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all closed curves $C \subset X$ passing through $x$. Here $L \cdot C$ denotes the intersection number while $\mathrm{mult}_x C$ denotes the multiplicity of the curve $C$ at $x$. To compute the Seshadri constant $\varepsilon(X, L, x)$, it suffices to take only irreducible and reduced curves $C$ in the above definition. Seshadri's criterion for ampleness of a line bundles says that $L$ is ample if and only if $\varepsilon(X, L, x) > 0$ for all $x \in X$.

Let $L$ be an ample line bundle on $X$. It is easy to see that

$$\varepsilon(X, L, x) \leq \sqrt[n]{L^n},$$

where $n$ is the dimension of the variety $X$ and $L^n$ is the top self-intersection number of $L$. One then defines

$$\varepsilon(X, L, 1) := \sup_{x \in X} \varepsilon(X, L, x).$$

Similarly, we have

$$\varepsilon(X, L) := \inf_{x \in X} \varepsilon(X, L, x).$$

Seshadri constants have many interesting applications and they are now the focus of a very active area of research. Some of the guiding problems on Seshadri constants involve computing Seshadri constants, giving bounds on them, checking if they are irrational, and interpolation problems. Computing Seshadri constants is frequently very difficult and usually it is only possible to give some bounds. In some special cases, however, it is

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possible to compute them exactly. In this paper we compute Seshadri constants of line bundles on Bott towers at all points.

Most of the existing work on Seshadri constants has been in the case of surfaces. Among the few cases in higher dimensions where Seshadri constants have been studied are abelian varieties (for example, see [Na, La, Ba, Deb]), toric varieties (for example, see [DiR, HMP, It1, It2]), Fano varieties (for example, see [BS, LZ]), and Grassmann bundles over curves ([BHNN]). For a survey of research around Seshadri constants, see [BDHKKSS].

In this paper, we study Seshadri constants for line bundles on Bott towers. We recall that Bott towers are special classes of toric varieties constructed iteratively as projective bundles of rank two vector bundles starting with the projective line $\mathbb{P}^1$. One can view them as a generalization of *Hirzebruch surfaces*, which are geometrically ruled surfaces over $\mathbb{P}^1$. See Section 2 for more details on Bott towers.

Seshadri constants of line bundles on Hirzebruch surfaces have been computed (see [Sy, Ga, HM]). In this paper we generalize this computation to an arbitrary Bott tower. Our main result (Theorem 3.1) computes the Seshadri constants for an arbitrary nef line bundle on a Bott tower at any point.

As noted above, Seshadri constants for line bundles on toric varieties have been studied by various authors. But for an arbitrary toric variety, Seshadri constants have been computed only for some classes of points, such as torus fixed points or points on the torus; see Remark 3.14 and Remark 3.16. In this paper, using the additional structure of a Bott tower, we compute Seshadri constants at arbitrary points.

In Section 2, we recall the construction of Bott towers and prove some properties which are used in Section 3. In Section 3, we prove our main theorem computing the Seshadri constants of nef line bundles on Bott towers. In Subsection 3.1, we include some remarks comparing our results with existing results in the literature and give examples illustrating our results.

**Notation.** We work over the field of complex numbers. We write $D_1 \sim_{\text{lin}} D_2$ (respectively, $D_1 \equiv D_2$) if the divisors $D_1, D_2$ are linearly equivalent (respectively, numerically equivalent). When the variety $X$ is clear from the context, the Seshadri constant $\varepsilon(X, L, x)$ is denoted simply by $\varepsilon(L, x)$.

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## 2. Bott towers

In this section, after recalling the construction of Bott towers along with some results about them, we prove some results about Bott towers that will be used in the computation of Seshadri constants.

Bott towers are a particular class of nonsingular projective toric varieties. They were constructed by Grossberg and Karshon (see [GK]). Grossberg and Karshon have also shown that Bott towers are degenerations of Bott-Samelson varieties, which are desingularizations of Schubert varieties.

For an integer $n \geq 0$, a *Bott tower of height* $n$

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = \{\text{point}\} \tag{2.1}$$

is defined inductively as an iterated $\mathbb{P}^1$–bundle so that at the $k$-th stage of the tower, $X_k$ is of the form $\mathbb{P}(\mathcal{O}_{X_{k-1}} \oplus \mathcal{L})$ for a line bundle $\mathcal{L}$ over $X_{k-1}$. So $X_1$ is isomorphic to $\mathbb{P}^1$, $X_2$ is a Hirzebruch surface and so on. A classical example is the product of projective lines, which arises when the line bundle $\mathcal{L}$ is trivial at every stage.

We call any stage $X_i$ of the tower $X_n$ in (2.1) also a Bott tower.

### 2.1. Fan structure of a Bott tower.

The multiplicative group $\mathbb{C} \setminus \{0\}$ will be denoted by $\mathbb{C}^*$. Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus. Define its character lattice

$$M := \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n$$

and the dual lattice $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Let $\Delta_n$ be a fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ which defines the toric variety $X_n$ under the action of the torus $T$. The set of edges of $\Delta_n$ will be denoted by $\Delta_n(1)$. Let $e_1, \cdots, e_n$ be the standard basis for $\mathbb{R}^n$. Consider the following vectors:

$$\begin{aligned}
v_1 &= e_1, \cdots, v_n = e_n, \\
v_{n+1} &= -e_1 + c_{1,2}e_2 + \ldots + c_{1,n}e_n, \\
&\vdots \\
v_{n+i} &= -e_i + c_{i,i+1}e_{i+1} + \ldots + c_{i,n}e_n, 1 \leq i < n, \\
v_{2n} &= -e_n.
\end{aligned} \tag{2.2}$$

The fan $\Delta_n$ of $X_n$ is complete, and it consists of these $2n$ edges and $2^n$ maximal cones of dimension $n$ generated by these edges such that no cone contains both the edges $v_i$ and $v_{n+i}$ for $i = 1, \cdots, n$. It follows that any $k$-th stage Bott tower arises from a collection of integers $\{c_{i,j}\}_{1 \leq i < j \leq n}$ as in (2.2). These integers are called the *Bott numbers* of the given Bott tower. In this paper we will restrict our attention to the case when the Bott numbers $\{c_{i,j}\}_{\{1 \leq i < j \leq n\}}$ are all positive integers.

### 2.2. Picard group of a Bott tower.

The following is recalled from [KD, Section 2.2].

Let $D_i$ denote the invariant prime divisor corresponding to the edge $v_{n+i}$, and let $D'_i$ denote the invariant prime divisor corresponding to the edge $v_i$ for $i = 1, \cdots, n$. We have the following relations:

$$D'_1 \sim_{\text{lin}} D_1, \; D'_i \sim_{\text{lin}} D_i - c_{1,i}D_1 - \ldots - c_{i-1,i}D_{i-1} \tag{2.3}$$

for $i = 2, \cdots, n$. The Picard group of the Bott tower is

$$\text{Pic}(X_n) = \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_n .$$

If $L$ is a line bundle on $X_n$ which is numerically equivalent to $a_1D_1 + \ldots + a_nD_n$ for some integers $a_1, \cdots, a_n$, then we write $L \equiv (a_1, \cdots, a_n)$.

Let $D = \sum_{i=1}^k a_iD_i$ be a Cartier divisor on $X_n$. Then $D$ is ample (respectively, nef) if and only if $a_i > 0$ (respectively, $a_i \geq 0$) for all $i = 1, \cdots, n$ (see [KD, Theorem 3.1.1, Corollary 3.1.2]).

2.3. **Quotient construction of a Bott tower.** We recall the quotient construction of Bott tower from [BP, Theorem 7.8.7]. The Bott tower $X_n$ can be obtained as the quotient $U_n /\!\!/ G_n$ of

$$U_n = \{(z_1, w_1, \cdots, z_n, w_n) \in \mathbb{C}^{2n} \mid |z_i|^2 + |w_i|^2 \neq 0, \; 1 \leq i \leq n\} \cong (\mathbb{C}^2 \setminus 0)^n$$

for the action of the group

$$G_n = \{(t_\rho)_{\rho\in\Delta_n(1)} \in (\mathbb{C}^*)^{\Delta_n(1)} \mid \prod_{\rho\in\Delta_n(1)} t_\rho^{\langle u_i, v_\rho\rangle} = 1\} \cong (\mathbb{C}^*)^n ,$$

where $u_1, \cdots, u_n$ is a basis of $M$. More explicitly, the inclusion $(\mathbb{C}^*)^n \hookrightarrow (\mathbb{C}^*)^{2n}$ is given by

$$(t_1, \cdots, t_n) \longmapsto (t_1, t_1, t_1^{-c_{1,2}}t_2, t_2, \cdots, t_1^{-c_{1,n}}t_2^{-c_{2,n}} \cdots t_{n-1}^{c_{n-1,n}}t_n, t_n).$$

A point of $X_n$ is denoted by the equivalence class $[z_1 : w_1 : \ldots : z_n : w_n]$. Note that $D'_i$ (respectively, $D_i$) is just the vanishing locus of the coordinate $z_i$ (respectively, $w_i$), i.e., $D'_i = \mathbb{V}(z_i)$ (respectively, $D_i = \mathbb{V}(w_i)$) for $1 \leq i \leq n$ (see [CLS, Example 5.2.5]).

We have

$$U_n \cong U_{n-1} \times (\mathbb{C}^2 \setminus 0), \quad (z_1, w_1, \cdots, z_n, w_n) \longmapsto ((z_1, w_1, \cdots, z_{n-1}, w_{n-1}) \times (z_n, w_n))$$

and

$$G_n \cong G_{n-1} \times \mathbb{C}^*, \quad (t_1, \cdots, t_n) \longmapsto ((t_1, \cdots, t_{n-1}), t_n),$$

where the last factor $t_n$ acts trivially on $U_{n-1}$. Thus $X_{n-1} = U_{n-1} /\!\!/ G_{n-1}$ is the Bott tower associated to the Bott numbers $\{c_{i,j}\}_{\{1\leq i<j\leq n-1\}}$. This also induces the map

$$X_n \longrightarrow X_{n-1}, \quad [z_1 : w_1 : \ldots, : z_n : w_n] \longmapsto [z_1 : w_1 : \ldots : z_{n-1} : w_{n-1}].$$

In general, for each $1 \leq i \leq n$, there is a map

$$X_i \longrightarrow X_{i-1}, \quad [z_1 : w_1 : \ldots : z_i : w_i] \longmapsto [z_1 : w_1 : \ldots : z_{i-1} : w_{i-1}]$$

together with a section given by

$$X_{i-1} \longrightarrow X_i, \quad [z_1 : w_1 : \ldots : z_{i-1} : w_{i-1}] \longmapsto [z_1 : w_1 : \ldots : z_{i-1} : w_{i-1} : 0 : 1] .$$

2.4. **Basic set-up.** Fix a point $x \in X_n$. Now we describe a special class of subvarieties $X_n^{(j)}$ of $X_n$ for $1 \leq j \leq n$ equipped with rational curves $\Gamma_n^{(j)} \subset X_n^{(j)}$. We emphasize that these subvarieties and rational curves *depend on the given point $x$*. However, for convenience, we omit indicating this in the notation.

Set $X_i^{(1)} := X_i$ for every $1 \leq i \leq n$. For every $2 \leq i \leq n$, let

$$\pi_i : X_i \longrightarrow X_1$$

be the composition of maps in (2.1). Define $X_i^{(2)} := \pi_i^{-1}(\pi_i(x))$. Note that $x \in X_n^{(2)}$.

Then $X_n^{(2)} \longrightarrow X_{n-1}^{(2)} \longrightarrow \cdots \longrightarrow X_2^{(2)}$ is a Bott tower (see Proposition 2.1 below). For every $3 \leq i \leq n$, let $\pi_{2,i} : X_i^{(2)} \longrightarrow X_2^{(2)}$ be the composition of these map. Define $X_i^{(3)} := \pi_{2,i}^{-1}(\pi_{2,i}(x))$.

Proceeding this way, we define $X_i^{(j)}$ for every $1 \leq j \leq i \leq n$. Note that $x \in X_n^{(j)}$ for all $1 \leq j \leq n$. Further, $X_i^{(i)} = \mathbb{P}^1$ for each $1 \leq i \leq n$. See Figure 1 below.

$$\begin{array}{ccccccccccccc}
\mathbb{P}^1 = X_n^{(n)} & \subset & X_n^{(n-1)} & \subset & X_n^{(n-2)} & \subset & \dots & \subset & X_n^{(3)} & \subset & X_n^{(2)} & \subset & X_n^{(1)} \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{P}^1 = X_{n-1}^{(n-1)} & \subset & X_{n-1}^{(n-2)} & \subset & \dots\dots & \subset & X_{n-1}^{(3)} & \subset & X_{n-1}^{(2)} & \subset & X_{n-1}^{(1)} \\
 & & & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & X_{n-2}^{(n-2)} & \subset & \dots\dots & \subset & X_{n-2}^{(3)} & \subset & X_{n-2}^{(2)} & \subset & X_{n-2}^{(1)} \\
 & & & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & & & & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & & & & & X_3^{(3)} & \subset & X_3^{(2)} & \subset & X_3^{(1)} \\
 & & & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & & & X_2^{(2)} & \subset & X_2^{(1)} \\
 & & & & & & & & & & & & \downarrow \\
 & & & & & & & & & & & & \mathbb{P}^1 = X_1^{(1)}
\end{array}$$

FIGURE 1. Construction of $X_i^{(j)}, 1 \leq j \leq i \leq n$

**Proposition 2.1.** *Each vertical tower in Figure 1 is a Bott tower with positive invariants.*

*Proof.* Fix a point $x = [z_1^0 : w_1^0 : \ldots : z_n^0 : w_n^0] \in X_n$. Then for $j \leq i$,

$$X_i^{(j)} = \{[z_1^0 : w_1^0 : \ldots : z_{j-1}^0 : w_{j-1}^0 : z_j : w_j : z_{j+1} : w_{j+1} : \ldots : z_i : w_i] \mid (z_l, w_l) \in \mathbb{C}^2 \setminus 0 \text{ for } j \leq l \leq i\} \subset X_i.$$

This can be identified with a Bott tower of dimension $i - j + 1$ with Bott numbers $\{c_{k,l}\}_{\{j \leq k < l \leq i\}}$ via the map

$$[z_1^0 : w_1^0 : \ldots : z_{j-1}^0 : w_{j-1}^0 : z_j : w_j : \ldots : z_i : w_i] \longmapsto [z_j : w_j : \ldots : z_i : w_i].$$

Similarly $X_{i-1}^{(j)}$ (provided $j \leq i-1$) can be identified with a Bott tower of dimension $i - j$ with Bott numbers $\{c_{k,l}\}_{\{j \leq k < l \leq i-1\}}$. Also note that the map $X_i^{(j)} \to X_{i-1}^{(j)}$ is defined by

$$[z_1^0 : w_1^0 : \ldots : z_{j-1}^0 : w_{j-1}^0 : z_j : w_j : \ldots : z_i : w_i] \longmapsto [z_1^0 : w_1^0 : \ldots : z_{j-1}^0 : w_{j-1}^0 : z_j : w_j : \ldots : z_{i-1} : w_{i-1}].$$

Thus each vertical tower in Figure 1 of the form

$$X_n^{(j)} \longrightarrow X_{n-1}^{(j)} \longrightarrow \ldots \longrightarrow X_{j+1}^{(j)} \longrightarrow X_j^{(j)}$$

is a Bott tower with Bott numbers $\{c_{k,l}\}_{\{j \leq k < l \leq n\}}$. Since all the Bott numbers were assumed to be positive, this completes the proof. □

**Proposition 2.2.** *Let $x \in X_n$ be a point. Then $X_n^{(2)} \equiv D_1$.*

*Proof.* Let $x = [z_1^0 : w_1^0 : \ldots : z_n^0 : w_n^0] \in X_n$ be as before. Then

$$X_n^{(2)} = \{[z_1^0 : w_1^0 : z_2 : w_2 : \ldots : z_n : w_n] \mid (z_i, w_i) \in \mathbb{C}^2 \setminus 0 \text{ for } 2 \leq i \leq n\}.$$

Note that if $z_1^0 = 0$, then $X_n^{(2)} = D_1' \sim_{\text{lin}} D_1$ (by (2.3)), and $X_n^{(2)} = D_1$ when $w_1^0 = 0$. So we can assume both $z_1^0, w_1^0$ are non-zero. Since $X_n^{(2)}$ is a divisor in $X_n$, we can write

$$X_n^{(2)} \equiv b_1 D_1 + \ldots + b_n D_n, \quad \text{with } b_1, \cdots, b_n \in \mathbb{Z}. \tag{2.4}$$

Recall that the $(n-1)$-dimensional cones in the fan correspond to invariant curves in the toric variety; let $V(\tau)$ denote the invariant curve corresponding to the $(n-1)$-dimensional cone $\tau$ in the fan. Now consider the curves $C_i = V(\tau_i)$ in $X_n$, where $\tau_i = \text{Cone}(v_1, \cdots, \widehat{v}_i, \cdots, v_n)$ are the $(n-1)$-dimensional cones in the fan $\Delta_n$. Since $D_i \cdot C_j = \delta_{ij}$ (see [CLS, Corollary 6.4.3, Proposition 6.4.4]), (2.4) gives we get that $b_i = X_n^{(2)} \cdot C_i$.

Let $i > 1$. Then $C_i = D_1' \cap \left(\cap_{j \neq 1, i} D_j'\right)$. This implies that $X_n^{(2)} \cap C_i = \emptyset$, i.e., $b_i = 0$. Now $X_n^{(2)} \cap C_1 = \{[z_1^0 : w_1^0 : 0 : 1 : \ldots : 0 : 1]\}$. So $X_n^{(2)} \cdot C_1 = 1$, i.e., $b_1 = 1$. Thus $X_n^{(2)} \equiv D_1$. □

Let $1 \leq i \leq n$. Let $X_n^{(i)} \longrightarrow X_{n-1}^{(i)} \longrightarrow \ldots \longrightarrow X_{i+1}^{(i)} \longrightarrow X_i^{(i)}$ be a vertical Bott tower in Figure 1. Then by the construction of projective bundles, there is a section map $X_{j-1}^{(i)} \longrightarrow X_j^{(i)}$ for every $i+1 \leq j \leq n$.

For each $1 \leq i \leq n$, let $\sigma_i : X_i^{(i)} \longrightarrow X_n^{(i)}$ be the composition of section maps. Define

$$\Gamma_n^{(i)} := \sigma_i(X_i^{(i)})\,. \tag{2.5}$$

We have $\Gamma_n^{(i)} \subset X_n^{(i)}$ for each $i$ and $\Gamma_n^{(n)} = X_n^{(n)}$. We denote $\Gamma_n^{(1)}$ also by $\Gamma_n$. See Figure 2 below.

$$\begin{array}{ccccccccccc}
X_n^{(n)} & \subset & X_n^{(n-1)} & \subset & \dots & \subset & X_n^{(3)} & \subset & X_n^{(2)} & \subset & X_n^{(1)} \\
\| & & \cup & & & & \cup & & \cup & & \cup \\
\Gamma_n^{(n)} & & \Gamma_n^{(n-1)} & & \dots & & \Gamma_n^{(3)} & & \Gamma_n^{(2)} & & \Gamma_n = \Gamma_n^{(1)} \\
\| & & \| & & & & \| & & \| & & \| \\
\sigma_n(X_n^{(n)}) & & \sigma_{n-1}(X_{n-1}^{(n-1)}) & & \dots & & \sigma_3(X_3^{(3)}) & & \sigma_2(X_2^{(2)}) & & \sigma_1(X_1^{(1)})
\end{array}$$

FIGURE 2. Construction of $\Gamma_n^{(i)}, 1 \leq i \leq n$

**Proposition 2.3.** *The curves $\Gamma_n, \Gamma_n^{(2)}, \cdots, \Gamma_n^{(n)}$ defined in (2.5) span $\overline{\mathrm{NE}}(X_n)$, and they are dual to $D_1, \cdots, D_n$.*

*Proof.* Fix a point $x = [z_1^0 : w_1^0 : \dots z_n^0 : w_n^0] \in X_n$. Then for $1 \leq i \leq n$,

$$\Gamma_n^{(i)} = \{[z_1^0 : w_1^0 : \dots : z_{i-1}^0 : w_{i-1}^0 : z_i : w_i : 0 : 1 : \dots : 0 : 1] \mid (z_i, w_i) \in \mathbb{C}^2 \setminus 0\} \subset X_n^{(i)}.$$

Now $D_1 \cap \Gamma_n^{(1)} = \{[1 : 0 : 0 : 1 : \dots : 0 : 1]\}$, and hence

$$D_1 \cdot \Gamma_n^{(1)} = 1.$$

For $1 < i \leq n$, we have

$$D_1 \cap \Gamma_n^{(i)} = \begin{cases} \emptyset, & \text{if } w_1^0 \neq 0, \\ \Gamma_n^{(i)}, & \text{if } w_1^0 = 0. \end{cases}$$

Thus $D_1 \cdot \Gamma_n^{(i)} = 0$ when $w_1^0 \neq 0$. But $D_1 \sim_{\text{lin}} D_1'$ by (2.3), and $D_1' \cap \Gamma_n^{(i)} = \emptyset$ when $w_1^0 = 0$ because $(z_1^0, w_1^0) \in \mathbb{C}^2 \setminus 0$. Therefore, $D_1 \cdot \Gamma_n^{(i)} = 0$ when $1 < i \leq n$. Thus for $1 \leq i \leq n$, we have

$$D_1 \cdot \Gamma_n^{(i)} = \delta_{i1}.$$

Fix $1 < j \leq n$, and assume that

$$D_k \cdot \Gamma_n^{(i)} = \delta_{ik} \tag{2.6}$$

whenever $1 \leq i \leq n$ and $1 \leq k < j$.

We will show that $D_j \cdot \Gamma_n^{(i)} = \delta_{ij}$ for $1 \leq i \leq n$. First note that

$$D_j \cap \Gamma_n^{(j)} = \{[z_1^0 : w_1^0 : \dots : z_{j-1}^0 : w_{j-1}^0 : 1 : 0 : 0 : 1 : \dots : 0 : 1]\}$$

and hence $D_j \cdot \Gamma_n^{(j)} = 1$. For $1 \leq i \leq n, i \neq j$, we have

$$D_j \cap \Gamma_n^{(i)} = \begin{cases} \emptyset, & \text{if } w_j^0 \neq 0, \\ \Gamma_n^{(i)}, & \text{if } w_j^0 = 0. \end{cases}$$

Thus for $w_j^0 \neq 0$, we have that $D_j \cdot \Gamma_n^{(i)} = 0$. On the other hand, if $w_j^0 = 0$, then $z_j^0 \neq 0$ and hence

$$D'_j \cdot \Gamma_n^{(i)} = 0. \tag{2.7}$$

Again from (2.3), we have that

$$D_j \sim_{\text{lin}} D'_j + c_{1,j}D_1 + \ldots + c_{j-1,j}D_{j-1}. \tag{2.8}$$

Note that $w_j^0 = 0$ is possible only if $j < i$, and then by (2.6), we have that

$$D_k \cdot \Gamma_n^{(i)} \;=\; 0 \tag{2.9}$$

holds for $1 \leq k < j$.

Then from (2.7), (2.8) and (2.9), it follows that $D_j \cdot \Gamma_n^{(i)} = 0$. Hence $D_j \cdot \Gamma_n^{(i)} = \delta_{ij}$ holds also for $1 \leq i \leq n$.

So $\Gamma_n, \Gamma_n^{(2)}, \cdots, \Gamma_n^{(n)}$ are dual to $D_1, \cdots, D_n$. This implies that they span $\overline{\text{NE}}(X_n)$, because $D_1, \cdots, D_n$ span the nef cone of $X_n$ and $\overline{\text{NE}}(X_n)$ is dual to the nef cone of $X_n$. □

Let $x \in X_n$, and let $C$ be an irreducible and reduced curve containing $x$. Assume that $x \notin \Gamma_n$. Then $C \neq \Gamma_n$. More generally, we have the following:

**Lemma 2.4.** *For any $2 \leq i \leq n$, if $x \in \Gamma_n^{(i)} \setminus \Gamma_n^{(i-1)}$, then $C \not\subset D'_i$.*

*Proof.* Write $x = [z_1^0 : w_1^0 : \ldots : z_n^0 : w_n^0] \in X_n$. The condition that $x \in \Gamma_n^{(i)}$ implies that $z_j^0 = 0$ for all $j \geq i+1$. Similarly $x \notin \Gamma_n^{(i-1)}$ implies that $z_j^0 \neq 0$ for some $j \geq i$. Thus $z_i^0 \neq 0$, i.e., $x \notin D'_i$. Since $x \in C$, it follows that $C \not\subset D'_i$. □

**Lemma 2.5.** *Let $L \equiv (a_1, \cdots, a_n) \in \text{Pic}(X_n)$. Then $L|_{X_n^{(i)}} \equiv (a_i, \cdots, a_n)$ whenever $2 \leq i \leq n$.*

*Proof.* Let $x = [z_1^0 : w_1^0 : \ldots : z_n^0 : w_n^0] \in X_n$ be as before. Note that

$$X_n^{(i)} = \left\{[z_1^0 : w_1^0 : \cdots : z_{i-1}^0 : w_{i-1}^0 : z_i : w_i : \cdots : z_n : w_n] \mid (z_l, w_l) \in \mathbb{C}^2 \setminus 0 \ \text{ for } i \leq l \leq n\right\}$$

is isomorphic to

$$\left\{[z'_1 : w'_1 : \cdots : z'_{n-i+1} : w'_{n-i+1}] \mid (z'_l, w'_l) \in \mathbb{C}^2 \setminus 0 \ \text{ for } 1 \leq l \leq n-i+1\right\},$$

which is a Bott tower of dimension $n-i+1$, where we have identified $z'_j$ (respectively, $w'_j$) with $z_{i-1+j}$ (respectively, $w_{i-1+j}$). Note that

$$\text{Pic}(X_n^{(i)}) \;=\; \mathbb{Z}D_1^{(i)} \oplus \cdots \oplus \mathbb{Z}D_{n-i+1}^{(i)},$$

where $D_j^{(i)} = \mathbb{V}(w'_j)$ for $j = 1, \cdots, n-i+1$.

Let $i \leq j \leq n$, and consider

$$\begin{aligned} D_j \cap X_n^{(i)} &= \{[z_1^0 : w_1^0 : \ldots : z_{i-1}^0 : w_{i-1}^0 : z_i : w_i : \ldots : z_n : w_n] \in X_n \mid w_j = 0\} \\ &= \mathbb{V}(w'_{j-i+1}) \subseteq X_n^{(i)} \\ &= D_{j-i+1}^{(i)}. \end{aligned}$$

Now using induction on $j$, we show that $D_j|_{X_n^{(i)}} = 0$ for $1 \leq j < i$. Note that

$$D_j \cap X_n^{(i)} = \begin{cases} \emptyset, & \text{if } w_j^0 \neq 0, \\ X_n^{(i)}, & \text{if } w_j^0 = 0. \end{cases} \tag{2.10}$$

First suppose that $j = 1$. From (2.10), if $w_1^0 \neq 0$ then $D_1|_{X_n^{(i)}} = 0$. So assume $w_1^0 = 0$. Then from (2.3), we have $D_1 \sim_{\text{lin}} D_1'$. Since $w_1^0 = 0$, we have $D_1' \cap X_n^{(i)} = \emptyset$, which shows that $D_1|_{X_n^{(i)}} = 0$. Now assume that $D_l|_{X_n^{(i)}} = 0$ for all $l < j$. Again from (2.10), if $w_j^0 \neq 0$ then $D_j|_{X_n^{(i)}} = 0$.

So let $w_j^0 = 0$. Then from (2.3),

$$D_j \sim_{\text{lin}} D_j' + c_{1,i}D_1 + \ldots + c_{j-1,j}D_{j-1}. \tag{2.11}$$

Clearly, $D_j' \cap X_n^{(i)} = \emptyset$ as $w_j^0 = 0$. Now from (2.11) together with the induction hypothesis, it follows that $D_j|_{X_n^{(i)}} = 0$. This completes the proof. □

## 3. Seshadri constants

In this section we prove our main theorem which determines the Seshadri constants of nef line bundles on Bott towers. We follow the notation developed in Section 2 about Bott towers.

Let $n$ be a positive integer. We consider Bott towers

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

of height $n$. Then $X_1 = \mathbb{P}^1$. If $L = \mathcal{O}_{\mathbb{P}^1}(a)$ is a nef line bundle on $\mathbb{P}^1$, then by convention, the Seshadri constants of $L$ are given by $\varepsilon(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a), x) = a$ for every $x \in \mathbb{P}^1$.

**Theorem 3.1.** *Let $n$ be a positive integer. Let*

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

*be a Bott tower with positive Bott numbers. Let $X_n^{(2)}$ be the subvariety of $X_n$ defined in Figure 1 and let $\Gamma_n \subset X_n$ be the rational curve defined in Figure 2. Let $L \equiv a_1D_1 + \ldots + a_nD_n$ be a nef line bundle on $X_n$. Take any $x \in X_n$. Then the Seshadri constants of $L$ are given by the following.*

$$\varepsilon(X, L, x) = \begin{cases} \min\{a_1, \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)\}, & \text{if } x \in \Gamma_n, \\ \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x), & \text{if } x \notin \Gamma_n. \end{cases}$$

**Remark 3.2.** In Section 2 we defined subvarieties $X_n^{(i)}$ of $X_n$ for every $i \leq n$; see Figure 1. So $X_n^{(i)}$ is not defined when $i > n$. When $n = 1$, we note that $X_1 = \Gamma_1$. Therefore, Theorem 3.1 is to be interpreted as follows when $n = 1$: Since $X_1^{(2)}$ is not defined, the theorem asserts that $\varepsilon(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a), x) = a$ for all $x \in \mathbb{P}^1$. This holds by the convention on Seshadri constants for nef line bundles on $\mathbb{P}^1$.

We will prove Theorem 3.1 in this section. We start with some propositions.

**Proposition 3.3.** *With the notation as in Theorem 3.1,*

$$\varepsilon(L, x) \geq \min\{a_1, \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)\}$$

*for all* $x \in X_n$.

*Proof.* Let $C \subset X_n$ be an irreducible and reduced curve such that $m := \text{mult}_x(C) > 0$. Write $C = p_1\Gamma_n + p_2\Gamma_n^{(2)} + \ldots + p_n\Gamma_n^{(n)}$, for some non-negative integers $p_1, \ldots, p_n$.

First suppose that $C \not\subset X_n^{(2)}$. Then, using Proposition 2.2, we have

$$C \cdot X_n^{(2)} = C \cdot D_1 \geq m(\text{mult}_x(D_1)) \geq m,$$

by Bézout's theorem. Proposition 2.3 implies $C \cdot D_1 = p_1$. So $p_1 \geq m$. Thus

$$\frac{L \cdot C}{m} = \frac{a_1p_1 + \ldots + a_np_n}{m} \geq a_1.$$

Next suppose that $C \subset X_n^{(2)}$. Then from the definition of $\varepsilon(L|_{X_n^{(2)}}, x)$ it follows that

$$\frac{L \cdot C}{m} = \frac{L|_{X_n^{(2)}} \cdot C}{m} \geq \varepsilon(L|_{X_n^{(2)}}, x).$$

Consequently, $\frac{L \cdot C}{m} \geq \min\{a_1, \varepsilon(L|_{X_n^{(2)}}, x)\}$ for all irreducible and reduced curves $C \subset X_n$. The proposition follows. □

**Proposition 3.4.** *For* $n > 0$, *let*

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

*be a Bott tower with positive Bott numbers. Let* $L \equiv a_1D_1 + \ldots + a_nD_n$ *be a nef line bundle on* $X_n$. *Take any* $x \in \Gamma_n := \Gamma_n^{(1)}$ *(see (2.5)). Then*

$$\varepsilon(L, x) = \min\{a_1, \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)\}.$$

*Proof.* We know that $\Gamma_n$ is a smooth rational curve. Since $x \in \Gamma_n$, we have $\varepsilon(L, x) \leq L \cdot \Gamma_n = a_1$.

On the other hand, we always have

$$\varepsilon(L, x) = \inf_{x \in C \subset X_n} \frac{L \cdot C}{\text{mult}_x C} \leq \inf_{x \in C \subset X_n^{(2)}} \frac{L \cdot C}{\text{mult}_x C} = \varepsilon(L|_{X_n^{(2)}}, x).$$

So $\varepsilon(L, x) \leq \min\{a_1, \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)\}$ and the proposition follows from Proposition 3.3. □

**Lemma 3.5.** *With the notation as in Theorem 3.1, suppose that*

$$\frac{L \cdot C}{\text{mult}_x C} \geq \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$$

*for all irreducible and reduced curves* $C \not\subset X_n^{(2)}$. *Then* $\varepsilon(L, x) = \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$.

*Proof.* For any irreducible and reduced curve $C \subset X_n^{(2)}$, we have

$$\frac{L \cdot C}{\mathrm{mult}_x C} \geq \varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x) \tag{3.1}$$

by the definition of Seshadri constants. So by the given hypothesis, (3.1) holds for all curves in $X_n$. So $\varepsilon(L,x) \geq \varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x)$. Since the opposite inequality $\varepsilon(L,x) \leq \varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x)$ holds, the lemma is proved. □

Next we consider the second case of Theorem 3.1. Note that if $x \notin \Gamma_n$, then $n \geq 2$.

**Proposition 3.6.** *Let $n \geq 2$ be an integer. Let*

$$X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

*be a Bott tower with positive Bott numbers. Let $L \equiv a_1 D_1 + \ldots + a_n D_n$ be a nef line bundle on $X_n$. Take $x \in X_n$ such that $x \notin \Gamma_n$. Then $\varepsilon(X_n,\, L, x) = \varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x)$.*

*Proof.* We prove this by using induction on $n$. First set $n = 2$. This implies that $X_2^{(2)} = \mathbb{P}^1$ and $L|_{X_2^{(2)}} = (a_2)$ by Lemma 2.5. Hence $\varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x) = a_2$. So we need to prove that $\varepsilon(X_2,\, L, x) = a_2$, under the assumption that $x \notin \Gamma_2$.

Let $C = p_1\Gamma_2 + p_2\Gamma_2^{(2)} \subset X_2$ be an irreducible and reduced curve for some non-negative integers $p_1, p_2$ such that $m := \mathrm{mult}_x(C) > 0$. Note that $x \in \Gamma_2^{(2)} = X_2^{(2)}$. Hence $x \in \Gamma_2^{(2)} \setminus \Gamma_2$ and we have $C \not\subset D_2'$ by Lemma 2.4. By (2.3),

$$D_2' \sim_{\mathrm{lin}} D_2 - c_{1,2} D_1\,,$$

where $c_{1,2}$ is a positive integer. Hence

$$0 \leq C \cdot D_2' = C \cdot (D_2 - c_{1,2} D_1) = p_2 - c_{1,2} p_1\,.$$

So $p_2 \geq p_1$.

Now suppose that $C \neq \Gamma_2^{(2)} = X_2^{(2)}$. Then Bézout's theorem and Proposition 2.2 together give that $m \leq C \cdot X_2^{(2)} = C \cdot D_1 = p_1$. Hence

$$\frac{L \cdot C}{m} = \frac{a_1 p_1 + a_2 p_2}{m} \geq a_2.$$

On the other hand, since $x \in \Gamma_2^{(2)}$, it contributes to the Seshadri constant of $L$ at $x$. So $\varepsilon(X_2, L, x) \leq L \cdot \Gamma_2^{(2)} = a_2$. Since $\frac{L \cdot C}{m} \geq a_2$ for every curve $C \neq \Gamma_2^{(2)}$, it follows that $\varepsilon(X_2, L, x) = a_2$.

Now assume that

- $n \geq 3$, and
- the proposition holds for all Bott towers of height at most $n-1$.

Note that $L|_{X_n^{(2)}} \equiv (a_2, \cdots, a_n)$ by Lemma 2.5. By the induction hypothesis and Proposition 3.4, we know that

$$\varepsilon(X_n^{(2)},\, L|_{X_n^{(2)}}, x) = \begin{cases} \min\{a_2, \varepsilon(X_n^{(3)},\, L|_{X_n^{(3)}}, x)\}, & \text{if } x \in \Gamma_n^{(2)}, \\ \varepsilon(X_n^{(3)}, L|_{X_n^{(3)}},\, x), & \text{if } x \notin \Gamma_n^{(2)}. \end{cases}$$

**Case 1**: $x \in \Gamma_n^{(2)}$.

In this case, $\varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x) = \min\{a_2, \varepsilon(X_n^{(3)}, L|_{X_n^{(3)}}, x)\} \leq a_2$.

**Lemma 3.7.** *Let $C \subset X_n$ be an irreducible and reduced curve such that $C \not\subset X_n^{(2)}$ and $m := mult_x(C) > 0$. Then $\frac{L \cdot C}{m} \geq a_2$.*

*Proof.* Write $C = p_1\Gamma_n + p_2\Gamma_n^{(2)} + \ldots + p_n\Gamma_n^{(n)}$ for some non-negative integers $p_1, \cdots, p_n$.

Since $x \in \Gamma_n^{(2)} \setminus \Gamma_n$, we have $C \not\subset D_2'$ by Lemma 2.4. From (2.3) we know that $D_2' \equiv D_2 - c_{1,2}D_1$ for a positive integer $c_{1,2}$. Hence $0 \leq C \cdot D_2' = p_2 - c_{1,2}p_1$. This in turn implies $p_2 \geq p_1$.

On the other hand, since $C \not\subset X_n^{(2)}$, we have $p_1 \geq m$ (see the proof of Proposition 3.3). So

$$\frac{L \cdot C}{m} = \frac{a_1p_1 + a_2p_2 + \ldots + a_np_n}{m} \geq a_2.$$

Hence the lemma is proved. □

So $\frac{L \cdot C}{m} \geq a_2 \geq \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$ for all curve $C \not\subset X_n^{(2)}$. Then, from Lemma 3.5 it follows that $\varepsilon(L, x) = \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$, as required.

**Case 2**: $x \notin \Gamma_n^{(2)}$.

In this case, $\varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x) = \varepsilon(X_n^{(3)}, L|_{X_n^{(3)}}, x)$. Now choose the smallest integer $i$ such that $x \in \Gamma_n^{(i)}$. Note that such an $i$ exists, since $x \in \Gamma_n^{(n)} = X_n^{(n)}$. It follows that $3 \leq i \leq n$ and $x \notin \Gamma_n^{(j)}$ for $j \leq i-1$.

By the induction hypothesis,

$$\varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x) = \varepsilon(X_n^{(3)}, L|_{X_n^{(3)}}, x) = \cdots = \varepsilon(X_n^{(i)}, L|_{X_n^{(i)}}, x).$$

Note that $L|_{X_n^{(i)}} = (a_i, \cdots, a_n)$ by Lemma 2.5. Again, by the induction hypothesis,

$$\varepsilon(X_n^{(i)}, L|_{X_n^{(i)}}, x) = \min\{a_i, \varepsilon(X_n^{(i+1)}, L|_{X_n^{(i+1)}}, x)\}, \text{ if } i \leq n-1$$
$$\text{and } \varepsilon(X_n^{(i)}, L|_{X_n^{(i)}}, x) = a_i, \text{ if } i = n.$$

In either case, we get that $\varepsilon(X_n^{(i)}, L|_{X_n^{(i)}}, x) \leq a_i$.

Hence $\varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x) = \cdots = \varepsilon(X_n^{(i)}, L|_{X_n^{(i)}}, x) \leq a_i$.

**Lemma 3.8.** *Let $C \subset X_n$ be an irreducible and reduced curve such that $C \not\subset X_n^{(2)}$ and $m := mult_x(C) > 0$. Then $\frac{L \cdot C}{m} \geq a_i$.*

*Proof.* Write $C = p_1\Gamma_n + p_2\Gamma_n^{(2)} + \ldots + p_n\Gamma_n^{(n)}$ for some non-negative integers $p_1, \cdots, p_n$.

Since $x \in \Gamma_n^{(i)} \setminus \Gamma_n^{(i-1)}$, we have $C \not\subset D_i'$ by Lemma 2.4. Further,

$$D_i' \equiv D_i - c_{1,i}D_1 - c_{2,i}D_2 - \ldots - c_{i-1,i}D_{i-1}$$

for positive integers $c_{1,i}, \cdots, c_{i-1,i}$. Hence

$$0 \leq C \cdot D_i' = p_i - c_{1,i}p_1 - \ldots - c_{i-1,i}p_{i-1}$$

and this gives $p_i \geq p_1$.

On the other hand, since $C \not\subset X_n^{(2)}$, we have that $p_1 \geq m$ (see the proof of Proposition 3.3). So

$$\frac{L \cdot C}{m} = \frac{a_1p_1 + a_2p_2 + \ldots + a_np_n}{m} \geq a_i.$$

Hence the lemma is proved. □

Consequently, $\frac{L\cdot C}{m} \geq a_i \geq \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$ for all curve $C \not\subset X_n^{(2)}$. Then, by Lemma 3.5 it follows that $\varepsilon(L, x) = \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$, as required. This completes the proof of the proposition. □

*Proof of Theorem 3.1.* The theorem follows immediately from Proposition 3.4 and Proposition 3.6. □

**Remark 3.9.** A Bott tower of height two is a Hirzebruch surface (a geometrically ruled surface over $\mathbb{P}^1$) and Seshadri constants for ample line bundles on such surfaces were computed in [Sy, Theorem 3.27] and [Ga, Theorem 4.1]. When $n = 2$, Theorem 3.1 recovers the results of [Sy, Ga].

**Corollary 3.10.** *For $n > 0$, let $X_n \to X_{n-1} \to \ldots \to X_2 \to X_1 \to X_0$ be a Bott tower with positive Bott numbers. Let $L \equiv a_1D_1 + \ldots + a_nD_n$ be a nef line bundle on $X_n$. Let $x \in X_n$. Then the following hold:*

(1) $\varepsilon(X_n, L, x) = \min\left\{L \cdot \Gamma_n^{(i)} \mid x \in \Gamma_n^{(i)}\right\}$. *In particular, $\varepsilon(X_n, L, x) = a_i$ for some $i$.*
(2) $\varepsilon(X_n, L, x) \geq \min\{a_1, \cdots, a_n\}$.
(3) $\varepsilon(X_n, L, x) \leq a_n$.

*Proof.* The statements in the corollary are derived from Theorem 3.1. If $x \notin \Gamma_n$, then $\varepsilon(X, L, x) = \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$ and the statements follow immediately from induction on $n$.

If $x \in \Gamma_n$, then $\varepsilon(X, L, x) = \min\{a_1, \varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)\}$. It is easy to see that the statements in the corollary hold for $\varepsilon(X_n, L, x)$ if they hold for $\varepsilon(X_n^{(2)}, L|_{X_n^{(2)}}, x)$. □

**Remark 3.11.** Let $L$ be any nef line bundle on a surface $X$ and let $x \in X$. If $\varepsilon(X, L, x) = \frac{L\cdot C}{\text{mult}_x C}$, then $C$ is said to be a *Seshadri curve* for $L$ at $x$. If $X_n$ is a Bott tower and $L$ is a nef line bundle on $X_n$, then the first statement of Corollary 3.10 shows that one of the curves $\Gamma_n, \Gamma_n^{(2)}, \cdots, \Gamma_n^{(n)}$ (defined in (2.5)) is a Seshadri curve for $L$ at any point $x \in X_n$.

**Corollary 3.12.** *For $n > 0$, let $X_n \to X_{n-1} \to \ldots \to X_2 \to X_1 \to X_0$ be a Bott tower with positive Bott numbers. Let $L \equiv a_1D_1 + \ldots + a_nD_n$ be a nef line bundle on $X_n$. Then*

(1) $\varepsilon(X_n, L) = \min\{a_1, \cdots, a_n\}$, *and*
(2) $\varepsilon(X_n, L, 1) = a_n$.

*Proof.* We first prove (1). Let $a_i = \min\{a_1, \cdots, a_n\}$. It is easy to choose a point $x \in X_n$ such that $x \in \Gamma_n^{(i)} \setminus \Gamma_n^{(i-1)}$; for example, we can take $x = [z_1^0 : w_1^0 : \ldots : z_n^0 : w_n^0]$ such that $z_i^0 \neq 0$ and $z_l^0 = 0, w_l^0 = 1$ for $l > i$. Then $x \notin \Gamma_n^{(j)}$ for any $j \leq i-1$. By Theorem 3.1,

$$\varepsilon(L, x) = \varepsilon(L|_{X_n^{(2)}}, x) = \ldots = \varepsilon(L|_{X_n^{(i)}}, x) = \min\{a_i, \varepsilon(L|_{X_n^{(i+1)}}, x)\} = a_i .$$

For the last equality, use Corollary 3.10(1) and $a_i = \min\{a_1, \cdots, a_n\}$. Now it follows (again, using Corollary 3.10(1)) that the smallest Seshadri constant of $L$ is $a_i$. Hence $\varepsilon(L, x) = a_i$.

To prove (2), choose a point $x \in X_n$ satisfying $x \notin \Gamma_n^{(n-1)}$. Then

$$\varepsilon(L, x) = \varepsilon(L|_{X_n^{(2)}}, x) = \ldots = \varepsilon(L|_{X_n^{(n)}}, x) = \varepsilon(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_n), x) = a_n .$$

Since all Seshadri constants are bounded above by $a_n$ and the value $a_n$ is achieved at some point, it follows that $\varepsilon(L, 1) = a_n$. □

**Remark 3.13.** From the proof of Corollary 3.12, we observe that $\varepsilon(L, 1) = \varepsilon(L, x) = a_n$ for any $x \notin \Gamma_n^{(n-1)}$. It is easy to see that for a general point $x \in X_n$, we have $x \notin \Gamma_n^{(n-1)}$. Thus $\varepsilon(L, 1)$ is achieved at general points of $X_n$. On the other hand, $\varepsilon(X, L)$ is achieved at *special* points of $X_n$; namely, points $x$ satisfying $\Gamma_n^{(i)} \setminus \Gamma_n^{(i-1)}$ if $a_i = \min\{a_1, \ldots, a_n\}$.

3.1. **Remarks and Examples.** Now we will compare our results with some other results on Seshadri constants on toric varieties in literature. We will then give some examples to illustrate our results.

**Remark 3.14.** Seshadri constants for line bundles on toric varieties at torus fixed points are investigated in [DiR] via generation of jets. The case of equivariant vector bundles on toric varieties is studied in [HMP]. At a torus fixed point $x$, the Seshadri constant of an equivariant vector bundle $E$ is computed via the restriction of $E$ to the invariant curves passing through $x$ (see [HMP, Proposition 3.2]). We show now that Theorem 3.1 recovers this result for line bundles. So for line bundles, Theorem 3.1 can be viewed as a generalization of these results for all points on a Bott tower.

Recall that there is one-one correspondence between the set of torus-fixed points in a smooth complete toric variety and the set of maximal cones. Let us denote the fixed point corresponding to a maximal cone $\sigma$ by $x_\sigma$. Let $L \equiv a_1 D_1 + \ldots + a_n D_n$ be a nef line bundle on $X_n$. By [HMP, Corollary 3.3], $\varepsilon(L) = \min_{x_\sigma} \varepsilon(L, x_\sigma)$, where the minimum varies over all maximal cones $\sigma$ in $\Delta_n$. Now consider the maximal cone $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$. Then by [BDHKKSS, Corollary 4.2.2], we have $\varepsilon(L, x_\sigma) = \min\{a_1, \ldots, a_n\}$, since the invariant curves passing through $x_\sigma$ are $V(\tau_i)$, where $\tau_i = \mathrm{Cone}(v_1, \ldots, \widehat{v}_i, \ldots, v_n)$ for $i = 1, \ldots, n$ (see also [HMP, Proposition 3.2]). Now consider a maximal cone $\sigma'$ other that $\sigma$. Let $C$ be an invariant curve passing through $x_{\sigma'}$. Then $C = V(\tau)$ for an $(n-1)$-dimensional cone $\tau$ of the form

$$\tau = \mathrm{Cone}(v_1, \ldots, \widehat{v}_{i_1}, \ldots, \widehat{v}_{i_r}, \ldots, v_n, v_{n+i_1}, v_{n+i_2}, \ldots, \widehat{v}_{n+i_j}, \ldots, v_{n+i_r})$$

for some $j = 1, \ldots, r$, where $r$ varies from 1 to $n$ and $D \cdot V(\tau) \geq \min\{a_1, \ldots, a_n\}$ (see [KD, proof of Theorem 3.1.1]). Thus $\varepsilon(L, x_{\sigma'}) \geq \min\{a_1, \ldots, a_n\}$. Hence $\varepsilon(L) = \min\{a_1, \ldots, a_n\}$, which agrees with Corollary 3.12 (1).

**Remark 3.15.** Theorem 3.1 shows that the Seshadri constants of ample line bundles on Bott towers are integers at all points. By [BDHKKSS, Corollary 4.2.2], this holds for torus fixed points on an arbitrary toric variety. Note however that Seshadri constants can be non-integral on an arbitrary toric variety; see [It2, Example 1.4]. This example describes an ample line bundle $L$ on a toric surface $X$ such that $L^2 = 3$ and $3/2 \leq \varepsilon(X, L, x) \leq \sqrt{3}$ for some point $x \in X$.

**Remark 3.16.** In [It2], the author gives bounds on Seshadri constants on an arbitrary toric variety at any point. In some cases, these bounds give exact values. To apply this in our situation, let $L \equiv a_1 D_1 + \ldots + a_n D_n$ be an ample line bundle on $X_n$. Let $x \in T$, the torus of $X_n$. By a repeated application of [It2, Theorem 3.6], it is possible to show that $\varepsilon(L, x) = a_n$. This is a special case of our results; for example, it follows from Corollary 3.10, since clearly $x \notin \Gamma_n^{(n-1)}$.

We now give some examples illustrating our main theorem. We use the same set-up as in Theorem 3.1. Note that in each example below Corollary 3.12 is verified.

**Example 3.17.** Let $L \equiv (1, 3, 8, 4) \in \text{Pic}(X_4)$ and $x \in X_4$. We repeatedly use Theorem 3.1 and Corollary 3.10 to compute the Seshadri constants of $L$. If $x \in \Gamma_4$ then $\varepsilon(L, x) = 1$. So assume now that $x \notin \Gamma_4$. Then $\varepsilon(L, x) = \varepsilon(L|_{X_4^{(2)}}, x)$. Note that $L|_{X_4^{(2)}} \equiv (3, 8, 4)$. If $x \in \Gamma_4^{(2)}$ then $\varepsilon(L, x) = 3$. Finally, if $x \notin \Gamma_4^{(2)}$, then $\varepsilon(L, x) = \varepsilon(L|_{X_4^{(3)}}, x) = 4$.

Thus

$$\varepsilon(L, x) = \begin{cases} 1, & \text{if } x \in \Gamma_4, \\ 3, & \text{if } x \notin \Gamma_4, x \in \Gamma_4^{(2)}, \\ 4, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(2)}. \end{cases}$$

**Example 3.18.** Let $L \equiv (1, 2, 3, 8) \in \text{Pic}(X_4)$ and $x \in X_4$. Repeatedly applying Theorem 3.1 and Corollary 3.10,

$$\varepsilon(L, x) = \begin{cases} 1, & \text{if } x \in \Gamma_4, \\ 2, & \text{if } x \notin \Gamma_4, x \in \Gamma_4^{(2)}, \\ 3, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(2)}, x \in \Gamma_4^{(3)}, \\ 8, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(2)}, x \notin \Gamma_4^{(3)}. \end{cases}$$

**Example 3.19.** Let $L \equiv (3, 6, 2, 7) \in \text{Pic}(X_4)$ and $x \in X_4$. Here note that $x \in \Gamma_4 \Rightarrow x \in \Gamma_4^{(2)} \Rightarrow x \in \Gamma_4^{(3)}$.

Then

$$\varepsilon(L, x) = \begin{cases} 2, & \text{if } x \in \Gamma_4, \\ 2, & \text{if } x \notin \Gamma_4, x \in \Gamma_4^{(3)}, \\ 7, & \text{if } x \notin \Gamma_4, x \notin \Gamma_4^{(3)}. \end{cases}$$

**Example 3.20.** Let $L \equiv (3, 6, 5, 7, 9) \in \text{Pic}(X_5)$ and $x \in X_5$. Here note that $x \in \Gamma_5 \Rightarrow x \in \Gamma_5^{(2)} \Rightarrow x \in \Gamma_5^{(3)} \Rightarrow x \in \Gamma_5^{(4)}$.

Then

$$\varepsilon(L,x) = \begin{cases} 3, & \text{if } x \in \Gamma_5, \\ 5, & \text{if } x \notin \Gamma_5, x \in \Gamma_5^{(2)}, \\ 5, & \text{if } x \notin \Gamma_5, x \notin \Gamma_5^{(2)}, x \in \Gamma_5^{(3)} \\ 7, & \text{if } x \notin \Gamma_5, x \notin \Gamma_5^{(2)}, x \notin \Gamma_5^{(3)}, x \in \Gamma_5^{(4)}, \\ 9, & \text{if } x \notin \Gamma_5, x \notin \Gamma_5^{(2)}, x \notin \Gamma_5^{(3)}, x \notin \Gamma_5^{(4)}. \end{cases}$$

## References


[Ba] T. Bauer, Seshadri constants and periods of polarized abelian varieties, *Math. Ann.* **312** (1998), no. 4, 607–623.

[BDHKKSS] T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek and T. Szemberg, A primer on Seshadri constants, *Interactions of classical and numerical algebraic geometry*, 33–70, Contemp. Math., 496, Amer. Math. Soc., Providence, RI, 2009.

[BS] T. Bauer and T. Szemberg, Seshadri constants and the generation of jets, *J. Pure Appl. Algebra* **213** (2009), 2134–2140.

[BHNN] I. Biswas, K. Hanumanthu, D. S. Nagaraj and P. E. Newstead, Seshadri constants and Grassmann bundles over curves, to appear in *Ann. Inst. Fourier.*

[BP] V. M. Buchstaber and T. E. Panov, *Toric topology*, Mathematical Surveys and Monographs, 204, American Mathematical Society, Providence, RI, 2015.

[CLS] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124, American Mathematical Society, Providence, RI, 2011.

[Deb] O. Debarre, Seshadri constants of abelian varieties, *The Fano Conference*, 379–394, Univ. Torino, Turin, 2004.

[Dem] J.-P. Demailly, Singular Hermitian metrics on positive line bundles, *Complex algebraic varieties* (Bayreuth, 1990), 87–104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[DiR] S. Di Rocco, Generation of $k$-jets on toric varieties, *Math. Zeit.* **231** (1999), 169–188.

[Ga] L. F. García, Seshadri constants on ruled surfaces: the rational and the elliptic cases, *Manuscripta Math.* **119** (2006), 483–505

[GK] M. Grossberg and Y. Karshon, Bott towers, complete integrability, and the extended character of representations, *Duke Math. Jour.* **76** (1994), 23–58.

[HM] K. Hanumanthu and A. Mukhopadhyay, Multi-point Seshadri constants on ruled surfaces, *Proc. Amer. Math. Soc.* **145** (2017), 5145–5155.

[HMP] M. Hering, M. Mustaţă and S. Payne, Positivity properties of toric vector bundles, *Ann. Inst. Fourier* **60** (2010), 607–640.

[It1] A. Ito, Okounkov bodies and Seshadri constants, *Adv. Math.* **241** (2013), 246–262.

[It2] A. Ito, Seshadri constants via toric degenerations, *Jour. Reine Angew. Math.* **695** (2014), 151–174.

[KD] B. Khan and J. Dasgupta, Toric vector bundles on Bott tower, *Bull. Sci. Math.* **155** (2019), 74–91.

[La] R. Lazarsfeld, Lengths of periods and Seshadri constants of abelian varieties, *Math. Res. Lett.* **3** (1996), 439–447.

[LZ] Y. Liu and Z. Zhuang, Characterization of projective spaces by Seshadri constants, *Math. Zeit.* **289** (2018), 25–38.

[Na] M. Nakamaye, Seshadri constants on abelian varieties, *Amer. Jour. Math.* **118** (1996), 621–635.

[Sy] W. Syzdek, *Seshadri constants and geometry of surfaces*, Ph.D thesis, 2005, University of Duisburg-Essen.

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