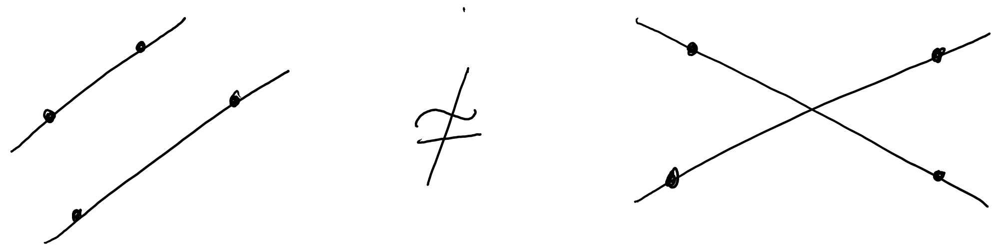
Non collinear points in the projective plane

joint w/ Ben O'Connor.

$p_1, \dots, p_n$ points in the plane such that no three are collinear.

Could think of plane $= \mathbb{R}^2$, but there are a couple of issues:

① Not all (pairs of) lines are "the same".



Avoid this for the moment by moving to the projective plane.

② A set of inequations. $(\det(\dots) \neq 0)$.

By choosing signs over $\mathbb{R}^2$; each component is defined by inequalities $\Rightarrow$ convex $\Rightarrow$ topologically boring.

So move to complex numbers.

$$X_n = \{(p_1, \dots, p_n) \in (\mathbb{CP}^2)^n \mid \text{no 3 are collinear}\}.$$

a line in $\mathbb{CP}^2$ is the solutions to linear equation, a copy of $\mathbb{CP}^1$.

What are the symmetries?

(1) Reordering/relabeling the points:

$S_n \circlearrowright (\mathbb{CP}^2)^n$ by permuting coordinates.

Maybe we care more about

$$Y_n := X_n / S_n \ ?$$

(2) Symmetries of $\mathbb{CP}^2$: $GL_3\mathbb{C}$ acts on $\mathbb{C}^3$ and descends to an action. Scalar matrices act by identify, so

$$PGL_3\mathbb{C} := GL_3\mathbb{C} / \mathbb{C}^\times \circlearrowright \mathbb{CP}^2.$$

Takes lines to lines, so acts on X_n.

Claim: The action of PGL_3 on X_4 is free (only identity has fixed points) and transitive (any point can be taken to any other).

For $PGL_2 \curvearrowright \mathbb{C}P^1$, this is about Möbius transforms being determined by 3 points.
Also generalizes to $n+2$ points "generic" points in $\mathbb{C}P^n$ corresponding to PGL_{n+1}.

<u>Proof</u> Given $(p_1, \dots, p_4), (q_1, \dots, q_4) \in X_4$,
want: $\exists! \; \mathbb{C}^\times A \in PGL_3$ s.t.

$$A p_i \in \mathbb{C}^\times q_i \qquad i = 1,2,3,4.$$

Think of $p_i, q_i \in \mathbb{C}^3$. Non collinear $\Rightarrow$ lin. ind.
WLOG, $(p_1, p_2, p_3, p_4) = (e_1, e_2, e_3, e_1+e_2+e_3)$

$e_1 = (1,0,0)$, $e_1+e_2+e_3 = (1,1,1)$.

$$q_4 = x q_1 + y q_2 + z q_3 \Rightarrow x, y, z \neq 0$$

Now take $A = \left(x q_1 \,\middle|\, y q_2 \,\middle|\, z q_3 \right)$. $\square$.

<u>Corollary</u> $X_4 \cong PGL_3\mathbb{C}$.

More generally, $X_n \cong (PGL_3\mathbb{C}) \times \underbrace{\left(X_n / PGL_3\mathbb{C} \right)}_{F_n}$

for $n \geq 4$.

$$F_n \cong \left\{ (p_5, \ldots, p_n) \,\middle|\, \begin{array}{c} (e_1, e_2, e_3, e_1+e_2+e_3, p_5, \ldots, p_n) \\ \in X_n \end{array} \right\}.$$

$$\left[\begin{array}{ll} \begin{tikzcd} PGL_3\mathbb{C} \arrow[r] & X_n \arrow[d] \\ & F_n \end{tikzcd} & \begin{array}{l} \text{is a principle } PGL_3\mathbb{C} \\ \text{bundle with a section.} \\ \text{The other projection} \cong \\ \left(\begin{array}{l} X_n \longrightarrow X_4 \cong PGL_3\mathbb{C} \\ (p_1, \ldots, p_n) \mapsto (p_1, p_2, p_3, p_4) \end{array} \right). \end{array} \end{array}\right]$$

<u>Theorem 1</u> (D-O'Connor) Explicit computation of $H^*(X_6; \mathbb{Q})$ with S_6 action.

(Also X_5, but this was known before).

<u>Theorem 2</u> (D-O'Connor)

$H^*(X_n; \mathbb{Q}) \cong H^*(PGL_3\mathbb{C}; \mathbb{Q}) \otimes H^*(F_n; \mathbb{Q})$

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as S_n-representations.

(By claim above, true as vector spaces)

$$H^*(X;\mathbb{Q}).$$

linearization of topology — ignoring torsion.

Connections:

- There is a unique smooth conic (solution set to quadratic) through each point of X_5.
  Conversely, 5 points on a conic C define a point of X_5.

  Since all conics are equivalent under $PGL_3\mathbb{C}$,

$$X_5/PGL_3\mathbb{C} \cong \{(p_1,\dots,p_5)\in C^5 \mid p_i \neq p_j\}/\mathrm{Aut}\,C.$$

$$\cong M_{0,5}$$ (moduli space of 5 pts on a genus 0 curve i.e. $\mathbb{P}^1 \cong C$)

Kisin-Lehrer for $\mathcal{M}_{0,n}$

- Blowing up $\mathbb{CP}^2$ at 6 generic points produces a smooth cubic surface

  $\hookrightarrow$ no 3 on a line, not all on a conic.

$$\begin{pmatrix}\text{moduli space of smooth} \\ \text{cubic surface}\end{pmatrix} \underset{\text{open}}{\subseteq} X_6 / PGL_3\mathbb{C},$$

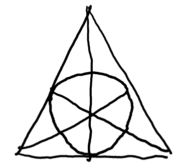
complement is $\mathcal{M}_{0,6}$.

The appropriate symmetry group for this space is $W(E_6)$, but the S_6 restriction can be obtained by our methods; see also Bergvall-Gounelas.

Similarly other $X_n / PGL_3\mathbb{C}$ for $n \leq 5$ are moduli spaces of del Pezzo surfaces of deg $(9-n)$.

- There are analogues of X_n and $Y_n = X_n / S_n$ defined over any field.

$\# Y_n(\mathbb{F}_q)$ is:

- polynomial for $n \leq 6$, elementary 
- quasi-polynomial for $n = 7, 8, 9$.

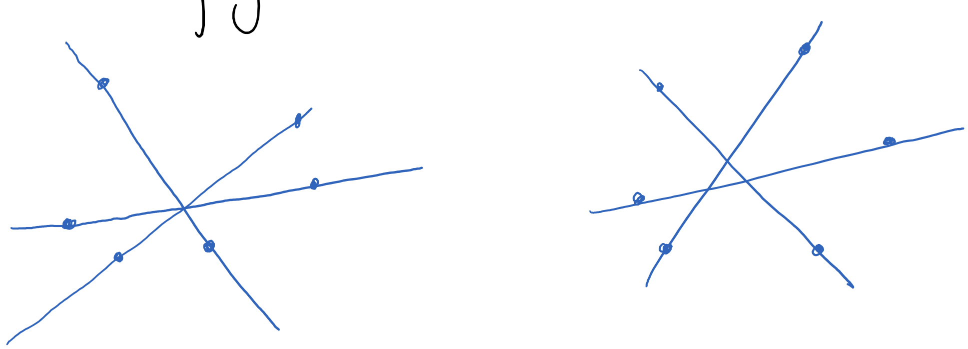
  (i.e. given by one of finitely many polynomials, chosen by q mod ...)

  elementary for 7, 8.

  $n = 9$ by Iampolskaia-Skorobogatov-Sorokin
- Not quasipolynomial for $n = 10$

  In progress work by IKKLPW.

- Not all configurations are the same:



X_6.

(More strata for higher n.)

$\rightsquigarrow$ Different choices for 7^{th} point given

⇝ Different choices for 7^{th} point given the first 6.

⇝ Topology of various strata for n large can be arbitrarily complicated

⇝ we expect them not to "cancel out" and affect the topology of X_n.

Mnëv's universality.

says any algebraic relation can be encoded in combinatorics of collinearity.

See also: Vakil, "Murphy's law ...".

Key Ideas:

Connection between $H^*(X(\mathbb{C}))$ and $\# X(\mathbb{F}_q)$ as a function of q.

Weil conjectures + comparison theorems + knowledge about classes.

Usual form: $H_c^*(X(\mathbb{C})) \rightsquigarrow \# X(\mathbb{F}_q)$.

In "good" situations, can reverse the arrow.

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- Why are our spaces, $X_n = X_n(\mathbb{C})$, "good"?

For $n \leq 6$,

$$\begin{array}{c} X_n \\ \downarrow \\ X_{n-1} \end{array}$$

is a fiber bundle.

The space of choices for the n^{th} point (fiber) depends continuously on the first $n-1$.

What is the fiber?

$\mathbb{P}^2 \setminus \bigcup \{$ lines joining each pair of the first $(n-1)$ pts $\}$.

This is a hyperplane complement, so

H^* is generated by classes pulled back

$H^*(\mathbb{C}^d \setminus \mathbb{C}^{d-1}) \simeq H^*(\mathbb{C} \setminus 0)$, which are "good" (the best one can expect, even).

(e.g. by Goresky-Macpherson).

- We also want to know the S_n action on $H^*(X_n; \mathbb{Q})$. This

$\mathbb{F}_q$). This corresponds to finer information than $\#X(\mathbb{F}_q)$, specifically, the action of Frob_q on points of $X_n(\overline{\mathbb{F}_{q^d}})$ that produce points of $(X_n/S_n)(\mathbb{F}_q)$.

$x \mapsto x^q$

What is $(X_n/S_n)(\mathbb{F}_q)$?

NOT: $X_n(\mathbb{F}_q)/S_n = \left(X_n(\overline{\mathbb{F}_q})^{\mathrm{Frob}_q}\right)/S_n$,

but $\left(X_n(\overline{\mathbb{F}_q})/S_n\right)^{\mathrm{Frob}_q}$

For comparison, x^2+1 is an $\mathbb{R}$-polynomial, so its set of roots $\{\pm i\}$ is defined over $\mathbb{R}$, but not the individual roots.

$$(\mathrm{Sym}^2 \mathbb{A}^1)(\mathbb{R}) = (\mathrm{Sym}^2 \mathbb{C})^{z \mapsto \bar{z}} \neq \mathrm{Sym}^2 \mathbb{R}$$

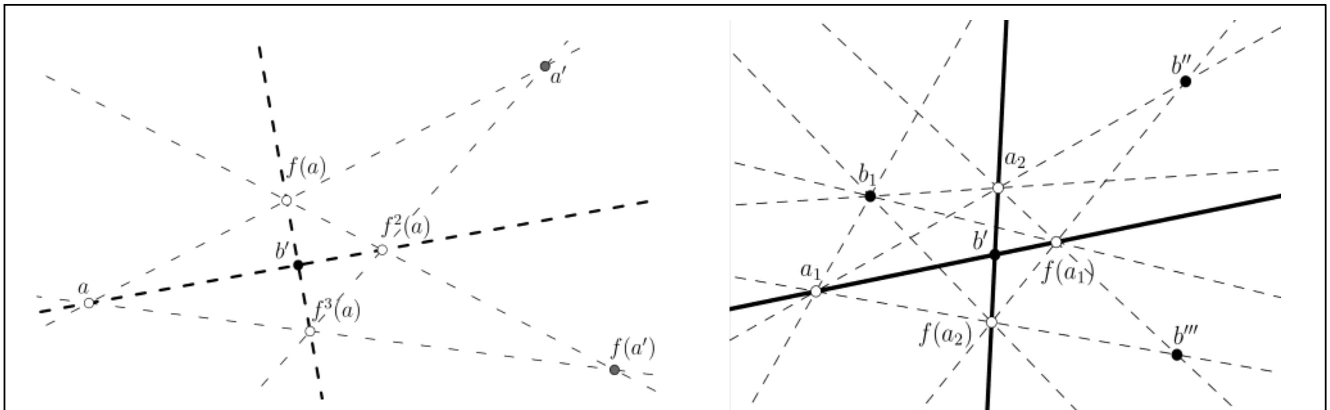
Since $\overline{\mathbb{F}_q} = \bigcup_{d \geq 1} \mathbb{F}_{q^d}$, any such orbit actually realized in some finite extension $X(\mathbb{F}_{q^d})$.

So we need to count points in $X_n(\overline{\mathbb{F}_q})$

So we need to count points in $X_n(\mathbb{F}_q)$ on which Frob_q acts by some element of S_n.

More precisely, we need to count them by conjugacy class in S_n, since the actual element depends on the choice of $\mathbb{F}_{q^d}$ (or rather $\overline{\mathbb{F}_q}$).

Combinatorics of points and lines in $\mathbb{P}^2(\mathbb{F}_{q^d})$



$(4,2)$ $(2,2,1,1)$

+ 9 more

+ "twisted Grothendieck-Lefschetz trace formula"
(i.e. for twisted/local coefficients)

$$\boxed{\sum_{p\in X(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Frob}_q \mid \mathcal{V}_p) = \sum_i \mathrm{Tr}(\mathrm{Frob}_q : H^{2n-i}_{\text{ét},c}(X;\mathcal{V})),}$$

+ character theory

= Theorem