Non-collinear points in the projective plane, joint w/ Ben O'Connor.

$p_1, \ldots, p_n$ points in the plane, such that no three are collinear.

Could think of plane = $\mathbb{R}^2$, but there are a couple of issues:

1. Not all (pairs of) lines are 'the same'.

   $\parallel \neq \not\parallel$

   Avoid this for the moment by moving to the projective plane.

2. A set of inequations $(\det(\ldots) \neq 0)$.

   By choosing signs over $\mathbb{R}^2$, each component is defined by inequalities $\Rightarrow$ convex $\Rightarrow$ topologically boring.
So move to complex numbers.

\[ X_n = \left\{ (p_1, \ldots, p_n) \in (\mathbb{CP}^2)^n \left| \text{no 3 are collinear} \right. \right\} \]

A line in \( \mathbb{CP}^2 \) is the solutions to linear equation, a copy of \( \mathbb{CP}^1 \).

What are the symmetries?

1. Reordering/relabeling the points:
   \( S_n \subseteq (\mathbb{CP}^2)^n \) by permuting coordinates.
   Maybe we care more about
   \[ Y_n := X_n / S_n ? \]

2. Symmetries of \( \mathbb{CP}^2 \): \( GL_3 \mathbb{C} \) acts on \( \mathbb{C}^3 \) and descends to an action. Scalar matrices act by identify, so
   \[ PGL_3 \mathbb{C} := GL_3 \mathbb{C} / \mathbb{C}^* \rightarrow \mathbb{CP}^2. \]
Takes lines to lines, so acts on \( X_n \).

Claim: The action of \( \text{PGL}_3 \) on \( X_4 \) is free (only identity has fixed points) and transitive (any point can be taken to any other).

For \( \text{PGL}_2 \subset \mathbb{C}^P \), this is about Möbius transforms being determined by 3 points. Also generalizes to \( n+2 \) points "generic" points in \( \mathbb{C}^P \) corresponding to \( \text{PGL}_{n+1} \).

**Proof** Given \( (p_1, \ldots, p_4), (q_1, \ldots, q_4) \in X_4 \), want: \( \exists! A \in \text{PGL}_3 \) s.t. 
\[
A p_i = C q_i \quad i = 1, 2, 3, 4.
\]

Think of \( p_i, q_i \in \mathbb{C}^3 \). Non-collinear \( \Rightarrow \) lin. ind.

Without loss of generality, \( (p_1, p_2, p_3, p_4) = (e_1, e_2, e_3, e_1 + e_2 + e_3) \) \((1, 1, 1)\).

\[
q_4 = xq_1 + yq_2 + zq_3 \Rightarrow x, y, z \neq 0
\]

Now take \( A = \begin{pmatrix} x & y & z \\ q_1 & q_2 & q_3 \end{pmatrix} \). \( \Box \).
Corollary \[ X_4 \cong \text{PGL}_3 \mathbb{C}. \]

More generally, \[ X_n = (\text{PGL}_3 \mathbb{C}) \times \left( \frac{\text{PGL}_3 \mathbb{C}}{F_n} \right) \]
for \( n \geq 4 \).

\[ F_n \cong \left\{ (p_5, \ldots, p_n) \mid (e_1, e_2, e_3, e_4 + e_2 + e_3, p_5, \ldots, p_n) \in X_n \right\}. \]

\[
\begin{bmatrix}
\text{PGL}_3 \mathbb{C} & \rightarrow & X_n \\
\downarrow & & \downarrow \\
F_n & \cong & (X_n \rightarrow X_4 \cong \text{PGL}_3 \mathbb{C})
\end{bmatrix}
\]

is a principle \( \text{PGL}_3 \mathbb{C} \)
bundle with a section.

The other projection \( \cong \)

\[
(X_n \rightarrow X_4 \cong \text{PGL}_3 \mathbb{C})
\]

\( (p_1, \ldots, p_n) \mapsto (p_1, p_2, p_3, p_4) \).

**Theorem 1** (D-O'Connor) Explicit computation of \( H^* (X_6; \mathbb{Q}) \) with \( S_6 \) action.

(Also \( X_5 \), but this was known before).

**Theorem 2** (D-O'Connor)

\[
h^* (\mathbb{C} \setminus \{0\}) \otimes H^* (F; \mathbb{Q})
\]
\[ H^*(X_n; \mathbb{Q}) \cong H^*(\text{PGL}_3 \mathbb{C}; \mathbb{Q}) \otimes H^*(F_n; \mathbb{Q}) \]

as \( S_n \)-representations.

(By claim above, true as vector spaces)

\[ H^*(X; \mathbb{Q}) \]

linearization of topology ignoring torsion.

Connections:

- There is a unique smooth conic (solution set to quadratic) through each point of \( X_5 \).
- Conversely, 5 points on a conic \( C \) define a point of \( X_5 \).

Since all conics are equivalent under \( \text{PGL}_3 \mathbb{C} \),

\[ X_5 / \text{PGL}_3 \mathbb{C} \cong \{(p_1, \ldots, p_5) \in \mathbb{C}^5 \mid p_i \neq p_j \} / \text{Aut } C \]

\[ \cong \mathcal{M}_{0,5} \text{ (moduli space of 5 pts} \]

on a genus 0 curve

i.e. \( \mathbb{P}^1 \cong C \)
Kisin-Lehrer for $M_{0,n}$

- Blowing up $\mathbb{P}^2$ at 6 generic points produces a smooth cubic surface.
  \[ \text{no 3 on a line, not all on a conic.} \]
  \[ \text{(moduli space of smooth cubic surface)} \subset \text{open } X_6 / \text{PGL}_3 \mathbb{C}. \]
  complement is $M_{0,6}$. The appropriate symmetry group for this space is $W(E_6)$, but the $S_6$ restriction can be obtained by our methods; see also Bergvall-Gounelas.

- Similarly other $X_n / \text{PGL}_3 \mathbb{C}$ for $n \leq 5$ are moduli spaces of del Pezzo surfaces of deg $(9-n)$.

- There are analogues of $X_n$ and $Y_n = X_n / S_n$ defined over any field.
\# \gamma_n(F_q) is:

- polynomial for \( n \leq 6 \), elementary
- quasi-polynomial for \( n = 7, 8, 9 \)

(i.e. given by one of finitely many polynomials, chosen by \( q \mod \ldots \))

elementary for \( 7, 8 \)

\( n = 9 \) by Iampolskaia, Skorobogatov, Sorokin

- Not quasi-polynomial for \( n = 10 \)

In progress work by IKKLPW

- Not all configurations are the same:

\[
\begin{array}{c}
\begin{array}{cc}
\text{X} & \text{X} \\
\end{array}

\end{array}
\]

(More strata for higher \( n \).

\[ \Rightarrow \text{Different choices for 7th point given} \]
Different choices for 7th point given the first 6.

Topology of various strata for $n$ large can be arbitrarily complicated so we expect them not to "cancel out" and affect the topology of $X_n$.

Mnev's universality says any algebraic relation can be encoded in combinatorics of collinearity. See also: Vakil, "Murphy's law ...".

Key Ideas:

Connection between $H^*(X(C))$ and $\# X(F_q)$ as a function of $q$.

Weil conjectures + comparison theorems + knowledge about classes.

Usual form: $H^*_c(X(C)) \sim \# X(F_q)$.

In "good" situations, can reverse the arrow.
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- Why are our spaces, $X_n = X_n(C)$, "good"?

  For $n \leq 6$, $X_n$ is a fiber bundle.

The space of choices for the $n^{th}$ point (fiber) depends continuously on the first $n-1$.

What is the fiber?

$\mathbb{P}^2 \setminus U \{ \text{lines joining each pair of the first } (n-1) \text{ pts} \}$.

This is a hyperplane complement, so $H^\ast$ is generated by classes pulled back $H^\ast(C \setminus C^{d-1}) = H^\ast(C \setminus 0)$, which are "good" (the best one can expect, even).

(e.g. by Goresky-MacPherson).

- We also want to know the $S_n$ action on $H^\ast(X : \mathbb{R})$. This is a key to finer information.
than \( \# X(F_q) \), specifically, the action of \( \text{Frob}_q \) on points of \( X_n(F_{q^d}) \) that produce points of \( (X_n/\Sigma_n)(F_q) \).

What is \( (X_n/\Sigma_n)(F_q) \)?

**NOT:** \( X_n(F_q) / \Sigma_n = (X_n(F_{\overline{q}})^{\text{Frob}_q}) / \Sigma_n \),

but \( (X_n(\overline{F_q})/\Sigma_n)^{\text{Frob}_q} \)

For comparison, \( x^2 + 1 \) is an \( \mathbb{R} \)-polynomial, so its set of roots \( \{ \pm i \} \) is defined over \( \mathbb{R} \), but not the individual roots.

\[
(Sym^2 A^1)(\mathbb{R}) = (Sym^2 \mathbb{C}^{2i \mapsto \overline{2}}) \neq Sym^2 \mathbb{R}
\]

Since \( \overline{F_q} = \bigcup_{d \geq 1} F_{q^d} \), any such orbit actually realized in some finite extension \( X(\overline{F_q}) \).

So we need to count points in \( X_n(\overline{F_q}) \)
So, we need to count points in $\mathbf{A}^n(\overline{\mathbb{F}}_q)$ on which Frobenius acts by some element of $S_n$.

More precisely, we need to count them by conjugacy class in $S_n$, since the actual element depends on the choice of $\overline{\mathbb{F}}_q$ (or rather, $\overline{\mathbb{F}}_p$).

Combinatorics of points and lines in $\mathbb{P}^2(\overline{\mathbb{F}}_q)$

(4,2) + 9 more

(2,2,1,1) + "twisted Grothendieck-Lefschetz trace formula"

(i.e. for twisted/local coefficients)

$$\sum_{p \in X(\mathbb{F}_q)} \text{Tr}(\text{Frob}_q | \mathcal{V}_p) = \sum_i \text{Tr}(\text{Frob}_q : H^{2n-i}_{\text{ét},c}(X; \mathcal{V})),$$
+ character theory

= Theorem