POSITIVITY OF LINE BUNDLES ON SPECIAL BLOW UPS OF $\mathbb{P}^2$

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ABSTRACT. Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree e. Let X be the blow up of $\mathbb{P}^2$ at r distinct smooth points $p_1, \ldots, p_r \in C$. Motivated by results in [10, 11, 7], we study line bundles on X and establish conditions for ampleness and k-very ampleness.

1. INTRODUCTION

Let X denote the blow up $\mathbb{P}^2$ at r points $p_1, \ldots, p_r \in \mathbb{P}^2$. It is interesting to ask when a given line bundle L on X has positivity properties such as ampleness, very ampleness, global generation, and more generally, k-*very ampleness* (see the definition below). If the points $p_1, \ldots, p_r$ are general in $\mathbb{P}^2$, this question has been extensively studied and is related to important conjectures in algebraic geometry. See [9, 13] for a detailed introduction and some results in this case.

There has also been some work on these questions when the points are special in some way. In [10, 11], Harbourne considered blow ups of $\mathbb{P}^2$ at r points (not necessarily distinct) on an irreducible and reduced plane cubic and a characterization of line bundles with various positivity properties (ample, global generated, effective, very ample) was given. De Volder [6] partially generalized results of [10, 11] by considering blow ups of points (not necessarily distinct) on a reduced and irreducible curve of degree $e \geqslant 4$ and giving sufficient conditions for global generation and very ampleness.

Let k be a non-negative integer. A line bundle L on a projective variety X is said to be k-*very ample* if the restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for all zero-dimensional subschemes $Z \subset X$ of length $k+1$; in other words, for all zero-dimensional subschemes Z such that $\dim(H^0(Z, \mathcal{O}_Z)) = k+1$.

Note that 0-very ampleness is equivalent to global generation and 1-very ampleness is equivalent to very ampleness. As a result, k-very ampleness is considered a more general positivity property for a line bundle. See [1, 2, 3] for more details on the notion of k-very ampleness.

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A general theorem for k-very ampleness on blow ups of projective varieties was proved by Beltrametti and Sommese in [4]. De Volder and Tutaj-Gasińska [7] study k-very ampleness for line bundles on blow ups of $\mathbb{P}^2$ at general points on an irreducible and reduced cubic. Szemberg and Tutaj-Gasińska [13] study k-very ampleness for line bundles on blow ups of $\mathbb{P}^2$ at general points. The property of k-very ampleness is also studied for other classes of surfaces as well as higher-dimensional varieties. See [5, 8], for instance.

Our primary motivation comes from [10, 11, 7]. These papers study positivity questions when X is the blow up of $\mathbb{P}^2$ at points on a plane cubic. In this paper we generalize some of these results by considering blow ups of $\mathbb{P}^2$ at r *distinct* and *smooth* points on an irreducible and reduced plane curve of degree e. More precisely, let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree e. Let $p_1, \ldots, p_r$ be distinct smooth points on C. We consider the blow up $\pi : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $p_1, \ldots, p_r$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ be the inverse images of $p_1, \ldots, p_r$ respectively. Given a line bundle $L = dH - m_1E_1 - m_2E_2 - \ldots - m_rE_r$ on X, we are concerned with conditions on $d, e, r, m_1, \ldots, m_r$ which ensure ampleness and k-very ampleness of L.

In Section 2, we study ampleness and prove our main result (Theorem 2.1) in this case. In Section 3, we study k-very ampleness. Here our main result is Theorem 3.6 which gives conditions for k-very ampleness.

We work throughout over the complex number field $\mathbb{C}$.

2. Ampleness

The following is our main theorem on ampleness.

Theorem 2.1. *Let C be an irreducible and reduced plane curve of degree e. Let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at r distinct smooth points $p_1, \ldots, p_r \in C$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ be the inverse images of $p_1, \ldots, p_r$ respectively. Let L be a line bundle on X with $L \cdot E_i > 0$ for all $1 \leqslant i \leqslant r$. Let C_1 denote the proper transform of C on X.*

If $L \cdot C_1 > 0$ and $L \cdot H > L \cdot (E_{i_1} + \ldots + E_{i_e})$ for any e distinct indices $i_1, \ldots, i_e \in \{1, \ldots, r\}$, then L is ample.

Proof. Let $d := L \cdot H > 0$ and $m_i := L \cdot E_i > 0$ for every $1 \leqslant i \leqslant r$. By permuting the points, if necessary, assume that $m_1 \geqslant \ldots \geqslant m_r$. So $L = dH - m_1E_1 - \ldots - m_rE_r$ and $d > m_1 + \ldots + m_e$, by hypothesis. Also, we have $C_1 = eH - E_1 - \ldots - E_r$.

We use the Nakai-Moishezon criterion for ampleness. We first verify that L meets all irreducible curves D on X positively. If $D = C_1$, then $L \cdot D > 0$ by hypothesis.

So assume that $D \neq C_1$. Write $D = fH - n_1E_1 - \ldots - n_rE_r$ with $f \geqslant 0$. If $f = 0$, then $D = -n_1E_1 - \ldots - n_rE_r$ is effective and this implies that $D = E_i$ for some i. Indeed, first note that not all n_i can be non-negative, because in that case D is negative of an effective curve. If $n_i < 0$ for some i, then $D \cdot E_i = n_i < 0$. So E_i is a component of D and hence $D = E_i$. By hypothesis, $L \cdot E_i = m_i > 0$.

Assume now that $f > 0$. Since $D \neq E_i$ for any i, it follows that $n_i \geqslant 0$ for all i. Since $D \neq C_1$, we have $ef \geqslant n_1 + \ldots + n_r$.

By hypothesis, $d > m_1 + \ldots + m_e$. Hence we have

$$\begin{aligned} df &> m_1 f + \ldots + m_e f \\ &= m_1 n_1 + \ldots + m_e n_e + m_1(f - n_1) + \ldots + m_e(f - n_e) \\ &\geqslant m_1 n_1 + \ldots + m_e n_e + m_{e+1}(ef - n_1 - n_2 - \ldots - n_e) \\ &\geqslant m_1 n_1 + \ldots + m_e n_e + m_{e+1}(n_{e+1} + \ldots + n_r) \\ &\geqslant m_1 n_1 + \ldots + m_e n_e + m_{e+1} n_{e+1} + m_{e+2} n_{e+2} + \ldots + m_r n_r. \end{aligned}$$

Hence $L \cdot D > 0$.

The condition $L^2 > 0$ follows now because of Proposition 2.2, which says that L is effective if $L \cdot H \geqslant L \cdot (E_1 + \ldots + E_e)$. □

Proposition 2.2. *With X as in Theorem 2.1, let $L = dH - m_1 E_1 - \ldots - m_r E_r$ be a line bundle on X with $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r$. If $d \geqslant m_1 + \ldots + m_e$, then L is effective.*

Proof. This proposition is completely analogous to [10, Lemma 1.4] and has the same proof. We give a proof for completeness.

We show that L can be written as a non-negative linear combination of line bundles of the form H, $H - E_1$, $2H - E_1 - E_2, \ldots, (e-1)H - E_1 - \ldots - E_{e-1}$, $eH - E_1 - \ldots - E_i$ for $i = e, e+1, \ldots, r$. Since all these are effective, so is L.

Let s be the largest index in $\{1, 2, \ldots, r\}$ such that $m_s \neq 0$. If $s \leqslant e - 1$, we have

$L = m_s(sH - E_1 - \ldots - E_s) + (m_{s-1} - m_s)((s-1)H - E_1 - \ldots - E_{s-1}) + \ldots + (m_1 - m_2)(H - E_1) + (d - m_1 - m_2 - \ldots - m_s)H.$

If $s \geqslant e$, let $L' = L - m_s(eH - E_1 - \ldots - E_s)$. Then it is easy to see that L' satisfies the hypotheses of the proposition and the value of s is smaller for it. So we are done by induction on s. □

Remark 2.3. In Theorem 2.1, the only hypothesis which is not in general necessary for ampleness is the condition that $L \cdot H > L \cdot (E_1 + \ldots + E_e)$. But this condition is necessary if e points among $p_1, \ldots, p_r$ are collinear (assume that $e < r$). For instance, consider a line l in $\mathbb{P}^2$ that meets C in e distinct smooth points, say $p_1, \ldots, p_e$. Then choose any other $r - e$ points. If L is ample, then L meets the proper transform of l positively. So we have $L \cdot H > L \cdot (E_1 + \ldots + E_e)$ after permuting the exceptional divisors, if necessary. Thus in order to be ample for *all* choices of r distinct smooth points on C, L must satisfy $L \cdot H > L \cdot (E_1 + \ldots + E_e)$ after a suitable permutation of $E_1, \ldots, E_r$.

In Corollary 2.4, we address the *uniform* case (i.e., $m_1 = \ldots = m_r = m$), where we can make more precise statements.

Corollary 2.4. *Let X be as in Theorem 2.1. Let $L = dH - m\sum_{i=1}^{r} E_i$ be a line bundle on X. If $r \geqslant e^2$, L is ample if and only if $L \cdot C_1 > 0$. If $r < e^2$ then L is ample if $d > em$.*

Proof. The proof is immediate from Theorem 2.1. Indeed, when $r \geqslant e^2$, the hypothesis gives $de > rm$ which implies $d > \frac{r}{e}m \geqslant m$. When $r < e^2$, the hypothesis gives $de > e^2m > rm$. □

We note that when $r < e^2$ the condition $d > em$ is also *necessary* if e of the points are collinear. See Remark 2.3.

Corollary 2.5. *With the set-up as in Theorem 2.1,* L *is nef if* $L \cdot C_1 \geqslant 0$ *and* $L \cdot H \geqslant L \cdot (E_1 + \ldots + E_e)$. *Further, if* $r \geqslant e^2$ *and* $L = dH - m\sum_{i=1}^{r} E_i$, *then* L *is nef if and only if* $L \cdot C_1 \geqslant 0$.

Proof. For nefness, we only need to check that $L \cdot D \geqslant 0$ for all effective curves. This is immediate from the proof of Theorem 2.1 and Corollary 2.4. □

Remark 2.6. In [10, 11], Harbourne considers the case $e = 3$. He defines a line bundle $L = dH - m_1E_1 - \ldots - m_rE_r$ to be *standard* with respect to the *exceptional configuration* $\{H, E_1, \ldots, E_r\}$ if $d \geqslant m_1 + m_2 + m_3$. Our hypothesis that $d \geqslant m_1 + \ldots + m_e$ may be considered as a generalization of the notion of standardness to the case of arbitrary e. Harbourne defines L to be *excellent* if it is standard and $L \cdot C_1 > 0$. Suppose that $m_i > 0$ for every i. One of the main results in [10, 11] says that L is ample *if and only if* it is excellent with respect to some exceptional configuration. See [10, 11] for more details.

Our main Theorem 2.1 may be considered as a generalization of one direction of this result to the case of arbitrary e. The converse is not true when $e \geqslant 4$. See Example 2.9.

Example 2.7. Take $e = 3$ and $r = 10$. Let $L = 8H - 3(E_1 + E_2 + E_3) - 2(E_4 + \ldots + E_{10})$. Consider the following transformation:

$H \mapsto 2H - E_1 - E_2 - E_3$, $E_1 \mapsto H - E_2 - E_3$, $E_2 \mapsto H - E_1 - E_3$, $E_3 \mapsto H - E_1 - E_2$, and $E_i \mapsto E_i$ for $i = 4, \ldots, 10$.

Under this, L is transformed to $7H - 2(E_1 + \ldots + E_{10})$. This is standard, in fact excellent, in the sense of [10, 11]. So L is ample. One can also check that L is ample as in Example 2.9. This example illustrates one of the main theorems in [10] which says that a line bundle L is ample if and only if $L \cdot C_1 > 0$ and L is standard with respect to *some* exceptional configuration.

Example 2.8. Let $e = 4$ and $r = 17$. So X is the blow up of $\mathbb{P}^2$ at 17 distinct smooth points on an irreducible, reduced plane quartic. Let $L = 11H - 3(E_1 + E_2 + \ldots + E_{13}) - (E_{14} + \ldots + E_{17})$. Then $L \cdot C_1 = 44 - 39 - 4 = 1$, but $L^2 = 121 - 117 - 4 = 0$. So L is not ample. Here note that $d = 11 < m_1 + m_2 + m_3 + m_4 = 12$. So the hypotheses in Theorem 2.1 can not be weakened.

Example 2.9. In this example, we show that the hypotheses of Theorem 2.1 are not always necessary for ampleness. Let C be an irreducible and reduced plane quartic and let $p_1, \ldots, p_{18}$ be distinct smooth points on C such that no four are collinear. Let $L = 10H - 3(E_1 + E_2 + E_3) - 2(E_4 + \ldots + E_{18})$. Since $d = 10 < 3 + 3 + 3 + 2 = 11$, the hypotheses of Theorem 2.1 are not satisfied. However, we claim that L is ample.

It is easy to check that $L^2 = 13$ and $L \cdot C_1 = 1$. So let $D \neq C_1$ be an irreducible and reduced curve. Write $D = fH - \sum_{i=1}^{18} n_iE_i$.

If $f = 0$, we may assume that $n_i \neq 0$ for some i. We claim in fact that $n_i \leqslant 0$ for all i. Since $D = -(\sum_i n_i E_i)$ is effective, n_i can not all be non-negative, as in that case D is negative of an effective divisor. So $n_i < 0$ for some i. Then $D \cdot E_i = n_i < 0$, so that E_i is a component of D. Subtracting E_i from D for all i with $n_i < 0$, we obtain an effective divisor of the form $\sum_{n_j > 0} (-n_j) E_j$, which must be the zero divisor. Hence $L \cdot D = \sum_i (-n_i) > 0$.

Now let $f \geqslant 1$ and $n_1, \ldots, n_{18} \geqslant 0$. In fact, if $f = 1$, since no four points are collinear, we have $L \cdot D > 0$. So let $f \geqslant 2$.

Since C_1 and D are distinct irreducible curves, $C_1 \cdot D = 4f - \sum_{i=1}^{18} n_i \geqslant 0$. Then $L \cdot D = 10f - 3(n_1+n_2+n_3) - 2(n_4+\ldots+n_{18}) = \left(8f - 2\sum_{i=1}^{18} n_i\right) + 2f - (n_1+n_2+n_3) \geqslant 2f - (n_1+n_2+n_3)$. If $n_1+n_2+n_3 = 0$, it follows that $L \cdot D > 0$. Otherwise, without loss of generality, let $n_1 > 0$. Intersecting D with the proper transforms of the line through p_1, p_2 and the line through p_1, p_3, we get $f \geqslant n_1+n_2$ and $f \geqslant n_1+n_3$. Hence $2f \geqslant 2n_1+n_2+n_3 > n_1+n_2+n_3$. The last inequality holds because $n_1 > 0$. Thus $L \cdot D > 0$.

Though $L = 10H - 3(E_1+E_2+E_3) - 2(E_4+\ldots+E_{18})$ in this example is ample, it is easy to see that L does not satisfy the condition $d \geqslant m_1 + m_2 + m_3 + m_4$ with respect to *any* exceptional configuration. This is easy to check by direct calculation. Note also that L is already standard in the sense of Harbourne [10, 11].

3. k-VERY AMPLENESS

Let $C, p_1, \ldots, p_r, X$ be as in Section 2. In this section we consider a line bundle $L = dH - m\sum_{i=1}^r E_i$ on X and investigate k-very ampleness of L for a non-negative integer k. We make the assumption that the number of points we blow up is large compared to e. Specifically, we assume that $r \geqslant e^2 + k + 1$.

First, we consider the question of global generation. In other words, we assume $k = 0$. Our arguments for $k = 0$ give a flavour of our arguments in the case $k \geqslant 1$, which we consider later.

We will use Reider's theorem [12, Theorem 1] which gives conditions for global generation and very ampleness. We only state the conditions for global generation below.

Theorem 3.1 (Reider). *Let X be a smooth complex surface and let N be a nef line bundle on X with $N^2 \geqslant 5$. If $K_X + N$ is not globally generated (here K_X is the canonical line bundle of X), then there exists an effective divisor D on X such that*

$$D \cdot N = 0, D^2 = -1, \text{ or } D \cdot N = 1, D^2 = 0.$$

The following is our theorem on global generation.

Theorem 3.2. *Let C be an irreducible and reduced plane curve of degree e and let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at r distinct smooth points $p_1, \ldots, p_r \in C$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and $E_1, \ldots, E_r$ the inverse images of $p_1, \ldots, p_r$ respectively. Let $L = dH - m\sum_{i=1}^r E_i$ be a line bundle on X with $m \geqslant 0$.*

If $(d+3)e > r(m+1)$ and $r \geqslant e^2 + 1$, then L is globally generated.

Proof. Since the conclusion holds if $m = 0$, we assume $m \geqslant 1$.

Let $N = L - K = (d+3)H - (m+1)\sum_{i=1}^{r} E_i$. If C_1 denotes the proper transform of C on X, then $N \cdot C_1 = (d+3)e - r(m+1) > 0$, by hypothesis. Moreover, $d+3 > \frac{r}{e}(m+1) > e(m+1)$, since $r > e^2$. So the hypotheses of Theorem 2.1 hold and N is ample.

Further, $N^2 = (d+3)^2 - r(m+1)^2 > \left(\frac{r^2}{e^2} - r\right)(m+1)^2 > (m+1)^2 \geqslant 4$. This is because $\frac{r^2}{e^2} - r \geqslant r(\frac{e^2+1}{e^2}) - r = r(1 + \frac{1}{e^2}) - r = \frac{r}{e^2} > 1$. So we can apply Reider's Theorem 3.1.

Suppose that L is not globally generated. By Theorem 3.1, there is an effective divisor D such that $D \cdot N = 1, D^2 = 0$. Since N is ample, this is the only possibility. Writing $D = fH - \sum_{i=1}^{r} n_i E_i$ and setting $n = \sum_{i=1}^{r} n_i$, we have

$$D \cdot N = (d+3)f - n(m+1) = 1 \text{ and } D^2 = f^2 - \sum_{i=1}^{r} n_i^2 = 0.$$

Note that if $n_i < 0$ for some i, then $D \cdot E_i < 0$. Thus E_i is a component of D and $D - E_i$ is effective. But then we get a contradiction because $1 = D \cdot N = N \cdot (D - E_i) + N \cdot E_i \geqslant m+1 \geqslant 2$ (since N is ample). Hence $n_i \geqslant 0$ for all i and $n = \sum_i n_i \geqslant 0$. Since $D^2 = 0$, in fact $n > 0$.

We consider two different cases: $n \geqslant r$ or $n < r$.

First, suppose that $n \geqslant r$. Since $f^2 = \sum_i n_i^2 \geqslant \frac{n^2}{r}$ and by hypothesis, $d + 3 > \frac{r}{e}(m+1)$, we have

$$1 = D \cdot N = (d+3)f - n(m+1) > \frac{n\sqrt{r}}{e}(m+1) - n(m+1). \tag{3.1}$$

We claim that $\frac{n\sqrt{r}}{e} - n \geqslant 1/2$ for any fixed e, all $r \geqslant e^2+1$ and for all $n \geqslant r$. Since this is a linear function in n of positive slope, it suffices to show that the function is non-negative when $n = r$. That is, we only have to show that $\frac{(\sqrt{r})r}{e} - r \geqslant 1/2$ for $r \geqslant e^2+1$. For a fixed e, this function is increasing for $r > 0$. So it suffices to show that $\frac{\sqrt{e^2+1}(e^2+1)}{e} - (e^2+1) \geqslant 1/2$. It is easy to see that this inequality holds for $e \geqslant 1$, for example by clearing the denominator and squaring.

Thus, by (3.1), $1 > \left(\frac{n\sqrt{r}}{e} - n\right)(m+1) \geqslant \frac{m+1}{2} \geqslant 1$, which is a contradiction.

Finally, we consider the case $n < r$. We have $f^2 = \sum_{i=1}^{r} n_i^2 \geqslant n$. So $1 = N \cdot D = (d+3)f - n(m+1) \geqslant \left(\frac{r\sqrt{n}}{e} - n\right)(m+1)$. We claim that $\frac{r\sqrt{n}}{e} - n \geqslant 1$, which as above leads to a contradiction.

We view $\frac{r\sqrt{n}}{e} - n$ as a quadratic in $\sqrt{n}$. Since the leading coefficient is -1, it is a downward sloping parabola. If we show that the value of this function is at least 1 for $n = 1$ and $n = r-1$, then it follows that the value of the function is at least 1 for all $1 \leqslant n \leqslant r-1$. This can be easily verified.

We conclude that L is globally generated. □

Now we will consider the case $k \geqslant 1$. Recall the criterion [1, Theorem 2.1] of Beltrametti-Francia-Sommese for k-very ampleness of L, which generalizes Reider's criterion in Theorem 3.1.

Theorem 3.3 (Beltrametti-Francia-Sommese)**.** *Let N be a line bundle on a surface X. Let $k \geqslant 0$ be an integer. Suppose that N is nef and $N^2 \geqslant 4k+5$. If $K_X + N$ is not k-very ample, then there exists an effective divisor D on X such that*

$$N \cdot D - k - 1 \leqslant D^2 < \frac{N \cdot D}{2} < k+1. \tag{3.2}$$

If (3.2) holds for an effective divisor D, then our next two results give some conditions that D must satisfy.

Proposition 3.4. *Let X be as in Theorem 3.2. Let $L = dH - m\sum_{i=1}^{r} E_i$ and $N = L - K_X = (d+3)H - (m+1)\sum_{i=1}^{r} E_i$ be line bundles on X. Let $k \geqslant 1$ be an integer. Suppose that $r \geqslant e^2+k+1$, $(d+3)e > r(m+1)$ and $m \geqslant k$. If an effective divisor $D = fH - \sum_{i=1}^{r} n_i E_i$ on X satisfies* (3.2)*, then $\sum_{i=1}^{r} n_i < r$.*

Proof. Set $n = \sum_{i=1}^{r} n_i$. Let $\alpha = N \cdot D = (d+3)f - (m+1)n$ and $\beta = D^2 = f^2 - \sum_{i=1}^{r} n_i^2$. If D satisfies (3.2), then we have

$$\alpha - k - 1 \leqslant \beta < \frac{\alpha}{2} < k+1. \tag{3.3}$$

By hypothesis, $(d+3)^2 > \frac{r^2}{e^2}(m+1)^2$. Since $(d+3)f = \alpha + (m+1)n$, we have

$$(d+3)^2 f^2 = (m+1)^2 n^2 + 2(m+1)n\alpha + \alpha^2 > \frac{r^2}{e^2}(m+1)^2\left(\beta + \sum_{i=1}^{r} n_i^2\right).$$

Since $r\sum_{i=1}^{r} n_i^2 \geqslant n^2$, we have

$$(m+1)^2 n^2 + 2(m+1)n\alpha + \alpha^2 > \frac{r^2}{e^2}(m+1)^2\beta + \frac{r}{e^2}(m+1)^2 n^2.$$

We now show that the above inequality is impossible for $n \geqslant r$. Specifically, we make the following claim.

Claim: Let r, e, k, α, β be as in the proposition and suppose that (3.3) holds. Then for $n \geqslant r$, we have

$$\frac{r^2}{e^2}(m+1)^2\beta + \frac{r}{e^2}(m+1)^2 n^2 \geqslant (m+1)^2 n^2 + 2(m+1)n\alpha + \alpha^2.$$

Proof of Claim: We consider the difference of the two terms in the required inequality as a quadratic function in n. Define

$$\lambda(n) := \left(\frac{r}{e^2} - 1\right)(m+1)^2 n^2 - 2(m+1)n\alpha + \frac{r^2\beta}{e^2}(m+1)^2 - \alpha^2.$$

This is quadratic in n with the leading coefficient $\left(\frac{r}{e^2} - 1\right)(m+1)^2 > 0$. We will show that $\lambda(r) \geqslant 0$ and $\lambda'(r) \geqslant 0$, which will prove the claim and the proposition.

$$
\begin{aligned}
\lambda(r) &= r^2(m+1)^2\left(\frac{r+\beta}{e^2}-1\right)-2(m+1)\alpha r-\alpha^2 \\
&\geqslant r^2(m+1)^2\left(\frac{e^2+k+1+\beta}{e^2}-1\right)-2(m+1)\alpha r-\alpha^2 \qquad (\text{since } r\geqslant e^2+k+1) \\
&= r^2(m+1)^2\left(\frac{k+1+\beta}{e^2}\right)-2(m+1)\alpha r-\alpha^2 \\
&\geqslant r(e^2+k+1)(m+1)^2\left(\frac{k+1+\beta}{e^2}\right)-2(m+1)\alpha r-\alpha^2 \\
&= r(m+1)^2(k+\beta+1)+\frac{r(k+1)(m+1)^2(k+\beta+1)}{e^2}-2(m+1)\alpha r-\alpha^2 \\
&\geqslant r(m+1)^2(k+\beta+1)+(k+1)(m+1)^2(k+\beta+1)-2(m+1)\alpha r-\alpha^2 \\
&\geqslant 0.
\end{aligned}
$$

The last inequality follows when we compare the first term with the third term and the second term with the fourth term. We use the inequalities $\alpha \leqslant \beta+k+1$ and $\alpha < 2(k+1)$ which hold by (3.3), and $m \geqslant k \geqslant 1$, which holds by hypothesis.

Next we show that $\lambda'(r) \geqslant 0$.

$\lambda'(n) = 2\left(\frac{r}{e^2}-1\right)(m+1)^2 n - 2(m+1)\alpha$. Thus

$$
\begin{aligned}
\lambda'(r) &= \left(\frac{2r^2}{e^2}-2r\right)(m+1)^2-2(m+1)\alpha \\
&\geqslant \frac{2r(e^2+k+1)}{e^2}(m+1)^2-2r(m+1)^2-2(m+1)\alpha \\
&= 2r(m+1)^2+2r(m+1)^2\frac{k+1}{e^2}-2r(m+1)^2-2(m+1)\alpha \\
&= 2r(m+1)^2\frac{k+1}{e^2}-2(m+1)\alpha \\
&\geqslant 2(m+1)^2(k+1)-2(m+1)\alpha \\
&\geqslant 0 \quad (\text{by (3.3) and the hypothesis that } m\geqslant k\geqslant 1).
\end{aligned}
$$

This completes the proof of the proposition. □

Proposition 3.5. *Let* $L = dH - m\sum_{i=1}^r E_i$ *and* $N = L - K_X = (d+3)H - (m+1)\sum_{i=1}^r E_i$. *Let* k *be a positive integer. Suppose that* $r \geqslant e^2+k+1$, $(d+3)e > r(m+1)$ *and* $m \geqslant k$. *Let* D *be an effective divisor on* X *such that* $D\cdot C_1 \geqslant 0$. *Then* D *does not satisfy* (3.2).

Proof. Let $D = fH - \sum_{i=1}^r n_iE_i$. Then $f \geqslant 0$ and if $f=0$, then $n_i \leqslant 0$ for all $i = 1,\dots,r$.

Let $n = \sum_{i=1}^r n_i$. Then $D^2 = f^2 - \sum_{i=1}^r n_i^2 \leqslant f^2 - n$. Indeed, this follows because $\sum_{i=1}^r n_i^2 \geqslant n$. Moreover, the assumption $D\cdot C_1 \geqslant 0$ implies that $ef \geqslant n$.

Suppose that D satisfies (3.2). We will obtain a contradiction.

First let $f = 0$. We have $N \cdot D = -(m+1)n < 2(k+1) \leqslant 2(m+1)$. So $n > -2$. On the other hand, $0 = ef \geqslant n$. So $n = 0$ or $n = -1$. Since each n_i is non-positive, if $n = 0$ then $D = 0$. If $n = -1$, then $D = E_i$ for some i. But then $N \cdot D = m+1 \geqslant k+1$, hence $N \cdot D - k - 1 \geqslant 0$, while $D^2 = -1$. This violates (3.2).

Let $f = 1$. In this case, we have $e \geqslant n$. We may also assume $n > 0$. Then $2(k+2) > N \cdot D = (d+3) - n(m+1) > \left(\frac{r}{e} - n\right)(m+1) \geqslant (e-n)(m+1) + \left(\frac{k+1}{e}\right)(m+1) \geqslant (e-n)(k+1) + \left(\frac{k+1}{e}\right)(k+1)$. Thus $0 \leqslant e - n < 2$. So $e = n$ or $e = n+1$.

Let $e = n+1$. Then $0 \geqslant 1 - n \geqslant D^2 \geqslant N \cdot D - k - 1 > k + 1 + \left(\frac{k+1}{e}\right)(k+1) - k - 1 > 0$, which is a contradiction. If $e = n$, then $0 \geqslant 1 - n \geqslant D^2 \geqslant N \cdot D - k - 1 > \left(\frac{k+1}{e}\right)(k+1) - k - 1$. Hence $k+1 < e$. But then $D^2 \leqslant 1 - n = 1 - e < -k \leqslant N \cdot D - k - 1$. The last inequality holds because N is ample and hence $N \cdot D > 0$. Again we have a contradiction, because this violates (3.2).

Now suppose that $f \geqslant 2$. As above, we have $2(k+1) > N \cdot D > \frac{rf(m+1)}{e} - n(m+1) \geqslant \frac{rf(m+1)}{e} - ef(m+1) = \left(\frac{rf}{e} - ef\right)(m+1) \geqslant \left(\frac{(k+1)f}{e}\right)(m+1)$.

$$\text{Thus } 2 > \frac{(m+1)f}{e}, \text{ or equivalently, } f < \frac{2e}{m+1}. \tag{3.4}$$

If $e \leqslant k+1$, then $f < \frac{2e}{m+1} \leqslant \frac{2e}{k+1} \leqslant 2$, which contradicts the hypothesis $f \geqslant 2$. So we may assume $e > k+1$.

We now make the following claim:

Claim: $f^2 - n \leqslant 2 - 2k$.

Proof of Claim: Note that $ef - n \leqslant 1$. Indeed, $(d+3)f > \frac{rf}{e}(m+1) > ef(m+1)$. Hence $2(k+1) > N \cdot D = (d+3)f - n(m+1) > (ef - n)(m+1) \geqslant (ef - n)(k+1)$. Thus $ef - n < 2$. On the other hand, by the hypothesis in the proposition $ef - n \geqslant 0$. Hence we have:

$$ef - n = 0 \text{ or } ef - n = 1. \tag{3.5}$$

On the other hand,

$$\begin{aligned}
f^2 - n &< f\left(\frac{2e}{m+1}\right) - n \quad \text{(by (3.4))} \\
&= ef + \frac{ef(1-m)}{m+1} - n \\
&\leqslant \frac{ef(1-k)}{k+1} + ef - n \quad (\text{ since } m \geqslant k) \\
&\leqslant \frac{ef(1-k)}{k+1} + 1 \quad (\text{ by (3.5)}) \\
&\leqslant f(1-k) + 1 \quad (\text{ since } e > k+1 \text{ and } 1-k \leqslant 0) \\
&\leqslant 2(1-k) + 1 = 3 - 2k \quad (\text{ since } f \geqslant 2 \text{ and } 1-k \leqslant 0).
\end{aligned}$$

This completes the proof of the claim. Now we consider three cases:

$\underline{k \geqslant 3}$: In this case, $f^2 - n \leqslant 2 - 2k \leqslant -k-1$. So $D^2 = f^2 - \sum_{i=1}^{r} n_i^2 \leqslant f^2 - n \leqslant -k-1$. But by (3.2), we have $N \cdot D - k - 1 \leqslant D^2$. Since N is ample, $-k \leqslant N \cdot D - k - 1$, contradicting the inequality $D^2 \leqslant -k-1$.

$\underline{k = 1}$: Then by (3.2), the claim, and the fact that $N \cdot D \geqslant 1$, we have $-1 \leqslant N \cdot D - 2 \leqslant f^2 - n \leqslant 0$. By (3.5), we have either $ef = n$ or $ef - 1 = n$.

If $ef = n$, then $0 \geqslant f^2 - n = f(f-e) \geqslant -1$. Since $f \geqslant 2$, the only possibility $f = e$. But this violates (3.4), because $m \geqslant k = 1$. On other hand, if $ef - 1 = n$, then $0 \geqslant f^2 - n = f^2 - ef + 1 = f(f-e) + 1 \geqslant -1$. Since $f \geqslant 2$, $f^2 - n$ must be -1. But then the only possibility is $f = 2$ and $e = 1$ and again we have a contradiction to (3.4).

$\underline{k = 2}$: By (3.2), the claim, and the fact that $N \cdot D \geqslant 1$, we have $-2 \leqslant N \cdot D - 3 \leqslant f^2 - n \leqslant -2$. Hence $f^2 - n = -2$. If $ef = n$, then $-2 = f^2 - ef = f(f-e)$. This contradicts (3.4). If $ef - 1 = n$, then $-2 = f(f-e) + 1$. Again we obtain a contradiction to (3.4).

This completes the proof of the proposition. □

Now we are ready to prove our main result on k-very ampleness.

Theorem 3.6. *Let C be an irreducible and reduced plane curve of degree e. Let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at r distinct smooth points $p_1, \ldots, p_r \in C$. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ be the inverse images of $p_1, \ldots, p_r$ respectively. Let k be a non-negative integer. Let $L = dH - m \sum_{i=1}^{r} E_i$ be a line bundle on X with $m \geqslant k$.*

If $(d+3)e > r(m+1)$ and $r \geqslant e^2 + k + 1$, then L is k-very ample.

Proof. When $k = 0$, this is the same as Theorem 3.2. So we will assume that $k \geqslant 1$ and use the criterion of Beltrametti-Francia-Sommese.

Let $N = L - K = (d+3)H - (m+1)\sum_{i=1}^{r} E_i$. Just as in the proof of Theorem 3.2, we conclude that N is ample.

Next we claim that $N^2 \geqslant 4k + 5$. Indeed, we have

$$\begin{aligned}
N^2 &= (d+3)^2 - r(m+1)^2 \\
&> \frac{r^2}{e^2}(m+1)^2 - r(m+1)^2 \\
&= (m+1)^2 r\left(\frac{r}{e^2} - 1\right) \\
&\geqslant (m+1)^2 r\left(\frac{k+1}{e^2}\right) \\
&> (m+1)^2(k+1) \\
&\geqslant 4(k+1) \qquad \text{(since } m \geqslant k \geqslant 1\text{)}.
\end{aligned}$$

Since N^2 is an integer and $N^2 > 4k+4$, we conclude that $N^2 \geqslant 4k+5$. Hence we can apply the criterion of Beltrametti-Francia-Sommese. Suppose that L is not k-very ample. Then there exists an effective divisor D on X such that (3.2) holds. We will obtain a contradiction.

Write $D = fH - n_1E_1 - \ldots - n_rE_r$ and let $n = \sum_i n_i$. By Proposition 3.4, we have $n < r$. By Proposition 3.5, $D \cdot C_1 < 0$. This implies that C_1 is a component of D and we may write $D = aC_1 + D'$ for a positive integer a and an effective divisor D'.

Write $D' = bH - \sum_{i=1}^r l_iE_i$ with $b \geqslant 0$. Then we have $f = ae + b$. So

$$\begin{aligned} 2(k+1) &> N \cdot D \\ &= (d+3)f - (m+1)n \\ &> \frac{r}{e}(ae+b)(m+1) - (r-1)(m+1) \quad \text{(since } n \leqslant r-1) \\ &= (r(a-1)+1)(m+1) + \frac{rb}{e}(m+1) \\ &\geqslant (r(a-1)+1)(k+1) + \frac{rb}{e}(k+1). \end{aligned}$$

Thus $a = 1$ and $b = 0$. In particular, $f = e$. We have $2(k+1) > N \cdot D = (d+3)e - n(m+1) > (r-n)(m+1)$. Hence $r - n < 2$. On the other hand, $r - n \geqslant 1$. So $r - n = 1$.

Since N is ample, $N \cdot D > 0$. Thus $-k \leqslant N \cdot D - k - 1 \leqslant D^2 = e^2 - \sum_{i=1}^r n_i^2 \leqslant e^2 - n = e^2 - r + 1 \leqslant -k$. The last inequality holds because $r \geqslant e^2 + k + 1$. Thus we have $D^2 = -k$ and $N \cdot D = 1$. But $N \cdot D > m + 1 \geqslant k + 1 > 1$. This is a contradiction.

The proof of the theorem is complete. □

Remark 3.7. [7, Theorem 4.1] gives conditions for k-very ampleness for any line bundle on the blow up of $\mathbb{P}^2$ at general points on an irreducible and reduced cubic. In our context, this is the case $e = 3$. If the line bundle is uniform, that is if $L = dH - m\sum_{i=1}^r E_i$ and if the number of points r is at least $10 + k$, then our Theorem 3.6 is comparable to this result. However, we note that [7, Theorem 4.1] deals with any (not just uniform) line bundle $L = dH - \sum_{i=1}^r m_iE_i$ and any $r \geqslant 3$.

The next two examples show that the hypothesis of Theorem 3.6 can not be weakened for $e = 3$.

Example 3.8. Let C be a smooth plane cubic. Let X be the blow up of 10 distinct points on C. Consider the line bundle $L = 7H - 2\sum_{i=1}^{10} E_i$ on X. We have $C_1 = 3H - \sum_{i=1}^{10} E_i$. By Corollary 2.4, L is ample. We use [2, Corollary 1.4] to show that L is not globally generated. According to this result, if a line bundle on a curve of positive genus is k-very ample, then the degree of the line bundle is at least k+2. In our example, if L is globally generated (that is, if it is 0-very ample), then $L_{|C}$ is also 0-very ample on C. But $\deg(L_{|C}) = L \cdot C = 1 < 2$. So the strict inequality, $e(d+3) > r(m+1)$, in the hypothesis of Theorem 3.6 can not be relaxed.

Example 3.9. This is a small variation on Example 3.8. Again let $e = 3, r = 10$, but now let $m = 7$. Consider $L = 24H - 7\sum_{i=1}^{10} E_i$. It is easy to check that L is ample (by Corollary 2.4) and globally generated (by Theorem 3.6). But L is not very ample because $L \cdot C_1 = 2 < 1+2$ (again by [2, Corollary 1.4]). Here the hypothesis in Theorem 3.6 on the number of points (namely, $r \geqslant e^2 + k + 1 = 10 + k$) does not hold.

Example 3.10. If $e > 3$, our hypotheses in Theorem 3.6 are not likely to be optimal. We will illustrate this with just one example.

Let $e = 5, k = 5, r = 31$ and consider $L_d = dH - 5(E_1 + \ldots + E_{31})$. By Theorem 3.6, L_{35} is 5-very ample. On the other hand, [2, Corollary 1.4] shows that L_{32} is *not* 5-very ample. We do not know if L_{33} or L_{34} is 5-very ample. Note that the criterion of Beltrametti-Francia-Sommese (Theorem 3.3) can not be applied here, since $N_d = L_d - K_X$ is not nef for $d < 35$. Indeed, $N_d \cdot C_1 = 5(d+3) - 186 < 0$, for $d < 35$. In other words, our method (which is to use Theorem 3.3 to show k-very ampleness) itself is not applicable to L_{33} and L_{34}.

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