POSITIVITY OF LINE BUNDLES ON SPECIAL BLOW UPS OF $\mathbb{P}^2$

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ABSTRACT. Let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree $e$. Let $X$ be the blow up of $\mathbb{P}^2$ at $r$ distinct smooth points $p_1, \ldots, p_r \in C$. Motivated by results in [10, 11, 7], we study line bundles on $X$ and establish conditions for ampleness and $k$-very ampleness.

1. INTRODUCTION

Let $X$ denote the blow up $\mathbb{P}^2$ at $r$ points $p_1, \ldots, p_r \in \mathbb{P}^2$. It is interesting to ask when a given line bundle $L$ on $X$ has positivity properties such as ampleness, very ampleness, global generation, and more generally, $k$-very ampleness (see the definition below). If the points $p_1, \ldots, p_r$ are general in $\mathbb{P}^2$, this question has been extensively studied and is related to important conjectures in algebraic geometry. See [9, 13] for a detailed introduction and some results in this case.

There has also been some work on these questions when the points are special in some way. In [10, 11], Harbourne considered blow ups of $\mathbb{P}^2$ at $r$ points (not necessarily distinct) on an irreducible and reduced plane cubic and a characterization of line bundles with various positivity properties (ample, global generated, effective, very ample) was given. De Volder [6] partially generalized results of [10, 11] by considering blow ups of points (not necessarily distinct) on a reduced and irreducible curve of degree $e \geq 4$ and giving sufficient conditions for global generation and very ampleness.

Let $k$ be a non-negative integer. A line bundle $L$ on a projective variety $X$ is said to be $k$-very ample if the restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_Z)$$

is surjective for all zero-dimensional subschemes $Z \subset X$ of length $k + 1$; in other words, for all zero-dimensional subschemes $Z$ such that $\dim(H^0(Z, \mathcal{O}_Z)) = k + 1$.

Note that 0-very ampleness is equivalent to global generation and 1-very ampleness is equivalent to very ampleness. As a result, $k$-very ampleness is considered a more general positivity property for a line bundle. See [1, 2, 3] for more details on the notion of $k$-very ampleness.

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A general theorem for k-very ampleness on blow ups of projective varieties was proved by Beltrametti and Sommese in [4]. De Volder and Tutaj-Gasińska [7] study k-very ampleness for line bundles on blow ups of $\mathbb{P}^2$ at general points on an irreducible and reduced cubic. Szemberg and Tutaj-Gasińska [13] study k-very ampleness for line bundles on blow ups of $\mathbb{P}^2$ at general points. The property of k-very ampleness is also studied for other classes of surfaces as well as higher-dimensional varieties. See [5, 8], for instance.

Our primary motivation comes from [10, 11, 7]. These papers study positivity questions when $X$ is the blow up of $\mathbb{P}^2$ at points on a plane cubic. In this paper we generalize some of these results by considering blow ups of $\mathbb{P}^2$ at $r$ distinct and smooth points on an irreducible and reduced plane curve of degree $e$. More precisely, let $C \subset \mathbb{P}^2$ be an irreducible and reduced curve of degree $e$. Let $p_1, \ldots, p_r$ be distinct smooth points on $C$. We consider the blow up $\pi : X \to \mathbb{P}^2$ at $p_1, \ldots, p_r$. Let $H$ denote the pull-back of $O_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ be the inverse images of $p_1, \ldots, p_r$ respectively. Given a line bundle $L = dH - m_1E_1 - m_2E_2 - \ldots - m_mE_r$ on $X$, we are concerned with conditions on $d, e, r, m_1, \ldots, m_r$ which ensure ampleness and k-very ampleness of $L$.

In Section 2 we study ampleness and prove our main result (Theorem 2.1) in this case. In Section 3 we study k-very ampleness. Here our main result is Theorem 3.6 which gives conditions for k-very ampleness.

We work throughout over the complex number field $\mathbb{C}$.

2. Ampleness

The following is our main theorem on ampleness.

**Theorem 2.1.** Let $C$ be an irreducible and reduced plane curve of degree $e$. Let $X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $r$ distinct smooth points $p_1, \ldots, p_r \in C$. Let $H$ denote the pull-back of $O_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ be the inverse images of $p_1, \ldots, p_r$ respectively. Let $L$ be a line bundle on $X$ with $L \cdot E_i > 0$ for all $1 \leq i \leq r$. Let $C_1$ denote the proper transform of $C$ on $X$.

If $L \cdot C_1 > 0$ and $L \cdot H > L \cdot (E_{i_1} + \ldots + E_{i_e})$ for any $e$ distinct indices $i_1, \ldots, i_e \in \{1, \ldots, r\}$, then $L$ is ample.

**Proof.** Let $d := L \cdot H > 0$ and $m_i := L \cdot E_i > 0$ for every $1 \leq i \leq r$. By permuting the points, if necessary, assume that $m_1 \geq \ldots \geq m_r$. So $L = dH - m_1E_1 - \ldots - m_mE_r$ and $d > m_1 + \ldots + m_e$, by hypothesis. Also, we have $C_1 = eH - E_1 - \ldots - E_r$.

We use the Nakai-Moishezon criterion for ampleness. We first verify that $L$ meets all irreducible curves $D$ on $X$ positively. If $D = C_1$, then $L \cdot D > 0$ by hypothesis.

So assume that $D \neq C_1$. Write $D = fH - n_1E_1 - \ldots - n_mE_r$ with $f \geq 0$. If $f = 0$, then $D = -n_1E_1 - \ldots - n_mE_r$ is effective and this implies that $D = E_i$ for some $i$. Indeed, first note that not all $n_i$ can be non-negative, because in that case $D$ is negative of an effective curve. If $n_i < 0$ for some $i$, then $D \cdot E_i = n_i < 0$. So $E_i$ is a component of $D$ and hence $D = E_i$. By hypothesis, $L \cdot E_i = m_i > 0$.
Assume now that \( f > 0 \). Since \( D \neq E_i \) for any \( i \), it follows that \( n_i \geq 0 \) for all \( i \). Since \( D \neq C_1 \), we have \( ef \geq n_1 + \ldots + n_r \).

By hypothesis, \( d \geq m_1 + \ldots + m_e \). Hence we have

\[
df > m_1 f + \ldots + m_e f = m_1 n_1 + \ldots + m_e n_e + m_1(f - n_1) + \ldots + m_e(f - n_e) \\
\geq m_1 n_1 + \ldots + m_e n_e + m_{e+1}(ef - n_1 - n_2 - \ldots - n_e) \\
\geq m_1 n_1 + \ldots + m_e n_e + m_{e+1}(n_{e+1} + \ldots + n_r) \\
\geq m_1 n_1 + \ldots + m_e n_e + m_{e+1}n_{e+1} + m_{e+2}n_{e+2} + \ldots + m_r n_r.
\]

Hence \( L \cdot D > 0 \).

The condition \( L^2 > 0 \) follows now because of Proposition 2.2, which says that \( L \) is effective if \( L \cdot H \geq L \cdot (E_1 + \ldots + E_e) \). \( \square \)

**Proposition 2.2.** With \( X \) as in Theorem 2.1, let \( L = dH - m_1 E_1 - \ldots - m_r E_r \) be a line bundle on \( X \) with \( m_1 \geq m_2 \geq \ldots \geq m_r \). If \( d \geq m_1 + \ldots + m_e \), then \( L \) is effective.

**Proof.** This proposition is completely analogous to \([10, \text{Lemma 1.4}]\) and has the same proof. We give a proof for completeness.

We show that \( L \) can be written as a non-negative linear combination of line bundles of the form \( H, H - E_1, 2H - E_1 - E_2, \ldots, (e - 1)H - E_1 - \ldots - E_{e-1}, eH - E_1 - \ldots - E_e \) for \( i = e, e + 1, \ldots, r \). Since all these are effective, so is \( L \).

Let \( s \) be the largest index in \( \{1, 2, \ldots, r\} \) such that \( n_s \neq 0 \). If \( s \leq e - 1 \), we have

\[
L = m_s(sH - E_1 - \ldots - E_s) + (m_{s-1} - m_s)((s-1)H - E_1 - \ldots - E_{s-1}) + \ldots + (m_1 - m_2)(H - E_1) + (d - m_1 - m_2 - \ldots - m_s)H.
\]

If \( s \geq e \), let \( L' = L - m_s(eH - E_1 - \ldots - E_s) \). Then it is easy to see that \( L' \) satisfies the hypotheses of the proposition and the value of \( s \) is smaller for it. So we are done by induction on \( s \). \( \square \)

**Remark 2.3.** In Theorem 2.1, the only hypothesis which is not in general necessary for ampleness is the condition that \( L \cdot H > L \cdot (E_1 + \ldots + E_e) \). But this condition is necessary if \( e \) points among \( p_1, \ldots, p_r \) are collinear (assume that \( e < r \)). For instance, consider a line \( l \) in \( \mathbb{P}^2 \) that meets \( C \) in \( e \) distinct smooth points, say \( p_1, \ldots, p_e \). Then choose any other \( r - e \) points. If \( L \) is ample, then \( L \) meets the proper transform of \( l \) positively. So we have \( L \cdot H > L \cdot (E_1 + \ldots + E_e) \) after permuting the exceptional divisors, if necessary. Thus in order to be ample for all choices of \( r \) distinct smooth points on \( C \), \( L \) must satisfy \( L \cdot H > L \cdot (E_1 + \ldots + E_e) \) after a suitable permutation of \( E_1, \ldots, E_r \).

In Corollary 2.4, we address the uniform case (i.e., \( m_1 = \ldots = m_r = m \)), where we can make more precise statements.

**Corollary 2.4.** Let \( X \) be as in Theorem 2.1. Let \( L = dH - m \sum_{i=1}^t E_i \) be a line bundle on \( X \). If \( r \geq e^2 \), \( L \) is ample if and only if \( L \cdot C_1 > 0 \). If \( r < e^2 \) then \( L \) is ample if \( d > em \).
Proof. The proof is immediate from Theorem 2.1. Indeed, when \( r \geq e^2 \), the hypothesis gives \( de > rm \) which implies \( d > \frac{r}{e}m \geq m \). When \( r < e^2 \), the hypothesis gives \( de > e^2m > rm \).

We note that when \( r < e^2 \) the condition \( d > em \) is also necessary if \( e \) of the points are collinear. See Remark 2.3.

**Corollary 2.5.** With the set-up as in Theorem 2.1, \( L \) is nef if \( L \cdot C_1 \geq 0 \) and \( L \cdot H \geq L \cdot (E_1 + \ldots + E_e) \). Further, if \( r \geq e^2 \) and \( L = dH - m \sum_{i=1}^r E_i \), then \( L \) is nef if and only if \( L \cdot C_1 > 0 \).

*Proof.* For nefness, we only need to check that \( L \cdot D \geq 0 \) for all effective curves. This is immediate from the proof of Theorem 2.1 and Corollary 2.4. \( \square \)

**Remark 2.6.** In [10, 11], Harbourne considers the case \( e = 3 \). He defines a line bundle \( L = dH - m_1 E_1 - \ldots - m_t E_t \) to be *standard* with respect to the *exceptional configuration* \( \{H, E_1, \ldots, E_t\} \) if \( d \geq m_1 + m_2 + m_3 \). Our hypothesis that \( d \geq m_1 + \ldots + m_e \) may be considered as a generalization of the notion of standardness to the case of arbitrary \( e \). Harbourne defines \( L \) to be *excellent* if it is standard and \( L \cdot C_1 > 0 \). Suppose that \( m_i > 0 \) for every \( i \). One of the main results in [10, 11] says that \( L \) is ample if and only if it is excellent with respect to some exceptional configuration. See [10, 11] for more details.

Our main Theorem 2.1 may be considered as a generalization of one direction of this result to the case of arbitrary \( e \). The converse is not true when \( e \geq 4 \). See Example 2.9.

**Example 2.7.** Take \( e = 3 \) and \( r = 10 \). Let \( L = 8H - 3(E_1 + E_2 + E_3) - 2(E_4 + \ldots + E_{10}) \). Consider the following transformation:

\[
H \mapsto 2H - E_1 - E_2 - E_3, \quad E_1 \mapsto H - E_2 - E_3, \quad E_2 \mapsto H - E_1 - E_3, \quad E_3 \mapsto H - E_1 - E_2, \quad E_i \mapsto E_i \text{ for } i = 4, \ldots, 10.
\]

Under this, \( L \) is transformed to \( 7H - 2(E_1 + \ldots + E_{10}) \). This is standard, in fact excellent, in the sense of [10, 11]. So \( L \) is ample. One can also check that \( L \) is ample as in Example 2.9. This example illustrates one of the main theorems in [10, 11] which says that a line bundle \( L \) is ample if and only if \( L \cdot C_1 > 0 \) and \( L \) is standard with respect to some exceptional configuration.

**Example 2.8.** Let \( e = 4 \) and \( r = 17 \). So \( X \) is the blow up of \( \mathbb{P}^2 \) at 17 distinct smooth points on an irreducible, reduced plane quartic. Let \( L = 11H - 3(E_1 + E_2 + \ldots + E_{13}) - (E_{14} + \ldots + E_{17}) \). Then \( L \cdot C_1 = 44 - 39 - 4 = 1 \), but \( L^2 = 121 - 117 - 4 = 0 \). So \( L \) is not ample. Here note that \( d = 11 < m_1 + m_2 + m_3 + m_4 = 12 \). So the hypotheses in Theorem 2.1 cannot be weakened.

**Example 2.9.** In this example, we show that the hypotheses of Theorem 2.1 are not always necessary for ampleness. Let \( C \) be an irreducible and reduced plane quartic and let \( p_1, \ldots, p_{18} \) be distinct smooth points on \( C \) such that no four are collinear. Let \( L = 10H - 3(E_1 + E_2 + E_3) - 2(E_4 + \ldots + E_{18}) \). Since \( d = 10 < 3 + 3 + 3 + 2 = 11 \), the hypotheses of Theorem 2.1 are not satisfied. However, we claim that \( L \) is ample.

It is easy to check that \( L^2 = 13 \) and \( L \cdot C_1 = 1 \). So let \( D \neq C_1 \) be an irreducible and reduced curve. Write \( D = fH - \sum_{i=1}^{18} n_i E_i \).
If \( f = 0 \), we may assume that \( n_i \neq 0 \) for some \( i \). We claim in fact that \( n_i \leq 0 \) for all \( i \). Since \( D = -(\sum_i n_i E_i) \) is effective, \( n_i \) can not all be non-negative, as in that case \( D \) is negative of an effective divisor. So \( n_i < 0 \) for some \( i \). Then \( D \cdot E_i = n_i < 0 \), so that \( E_i \) is a component of \( D \). Subtracting \( E_i \) from \( D \) for all \( i \) with \( n_i < 0 \), we obtain an effective divisor of the form \( \sum_{n_i > 0} (-n_i)E_i \), which must be the zero divisor. Hence \( L \cdot D = \sum_n (-n_i) > 0 \).

Now let \( f \geq 1 \) and \( n_1, \ldots, n_{18} \geq 0 \). In fact, if \( f = 1 \), since no four points are collinear, we have \( L \cdot D > 0 \). So let \( f \geq 2 \).

Since \( C_1 \) and \( D \) are distinct irreducible curves, \( C_1 \cdot D = 4f - \sum_{i=1}^{18} n_i \geq 0 \). Then \( L \cdot D = 10f - 3(n_1 + n_2 + n_3) - 2(n_4 + \ldots + n_{18}) = (8f - 2\sum_{i=1}^{18} n_i) + 2f - (n_1 + n_2 + n_3) \geq 2f - (n_1 + n_2 + n_3) \). If \( n_1 + n_2 + n_3 = 0 \), it follows that \( L \cdot D > 0 \). Otherwise, without loss of generality, let \( n_1 > 0 \). Intersecting \( D \) with the proper transforms of the line through \( p_1, p_2 \) and the line through \( p_1, p_3 \), we get \( f \geq n_1 + n_2 \) and \( f \geq n_1 + n_3 \). Hence \( 2f \geq 2n_1 + n_2 + n_3 > n_1 + n_2 + n_3 \). The last inequality holds because \( n_1 > 0 \). Thus \( L \cdot D > 0 \).

Though \( L = 10H - 3(E_1 + E_2 + E_3) - 2(E_4 + \ldots + E_{18}) \) in this example is ample, it is easy to see that \( L \) does not satisfy the condition \( d \geq m_1 + m_2 + m_3 + m_4 \) with respect to any exceptional configuration. This is easy to check by direct calculation. Note also that \( L \) is already standard in the sense of Harbourne \([10, 11]\).

### 3. \( k \)-very ampleness

Let \( C, p_1, \ldots, p_r, X \) be as in Section 2. In this section we consider a line bundle \( L = dH - m \sum_{i=1}^{r} E_i \) on \( X \) and investigate \( k \)-very ampleness of \( L \) for a non-negative integer \( k \). We make the assumption that the number of points we blow up is large compared to \( e \). Specifically, we assume that \( r \geq e^2 + k + 1 \).

First, we consider the question of global generation. In other words, we assume \( k = 0 \). Our arguments for \( k = 0 \) give a flavour of our arguments in the case \( k \geq 1 \), which we consider later.

We will use Reider’s theorem \([12\text{, Theorem 1}] \) which gives conditions for global generation and very ampleness. We only state the conditions for global generation below.

**Theorem 3.1** (Reider). *Let \( X \) be a smooth complex surface and let \( N \) be a nef line bundle on \( X \) with \( N^2 \geq 5 \). If \( K_X + N \) is not globally generated (here \( K_X \) is the canonical line bundle of \( X \)), then there exists an effective divisor \( D \) on \( X \) such that \( D \cdot N = 0, D^2 = -1 \), or \( D \cdot N = 1, D^2 = 0 \).*

The following is our theorem on global generation.

**Theorem 3.2.** *Let \( C \) be an irreducible and reduced plane curve of degree \( e \) and let \( X \to \mathbb{P}^2 \) be the blow up of \( \mathbb{P}^2 \) at \( r \) distinct smooth points \( p_1, \ldots, p_r \in C \). Let \( H \) denote the pull-back of \( O_{\mathbb{P}^2}(1) \) and \( E_1, \ldots, E_r \) the inverse images of \( p_1, \ldots, p_r \) respectively. Let \( L = dH - m \sum_{i=1}^{r} E_i \) be a line bundle on \( X \) with \( m \geq 0 \).

If \((d + 3)e > r(m + 1)\) and \( r \geq e^2 + 1 \), then \( L \) is globally generated.*
Proof. Since the conclusion holds if \( m = 0 \), we assume \( m \geq 1 \).

Let \( N = L - K = (d+3)H - (m+1) \sum_{i=1}^{r} E_i \). If \( C_1 \) denotes the proper transform of \( C \) on \( X \), then \( N \cdot C_1 = (d+3)e - r(m+1) > 0 \), by hypothesis. Moreover, \( d+3 > \frac{r}{e}(m+1) > e(m+1) \), since \( r > e^2 \). So the hypotheses of Theorem 2.1 hold and \( N \) is ample.

Further, \( N^2 = (d+3)^2 - r(m+1)^2 > \left( \frac{r^2}{e^2} - r \right) (m+1)^2 > (m+1)^2 \geq 4 \). This is because \( \frac{r^2}{e^2} - r \geq r(e^2 + 1) - r = r(1 - \frac{1}{e^2}) > \frac{r}{e^2} > 1 \). So we can apply Reider’s Theorem 3.1.

Suppose that \( L \) is not globally generated. By Theorem 3.1 there is an effective divisor \( D \) such that \( D \cdot N = 1, D^2 = 0 \). Since \( N \) is ample, this is the only possibility. Writing \( D = fH - \sum_{i=1}^{r} n_i E_i \) and setting \( n = \sum_{i=1}^{r} n_i \), we have

\[
D \cdot N = (d+3)f - n(m+1) = 1 \quad \text{and} \quad D^2 = f^2 - \sum_{i=1}^{r} n_i^2 = 0.
\]

Note that if \( n_i < 0 \) for some \( i \), then \( D \cdot E_i < 0 \). Thus \( E_i \) is a component of \( D \) and \( D - E_i \) is effective. But then we get a contradiction because \( 1 = D \cdot N = N \cdot (D - E_i) + N \cdot E_i \geq m+1 \geq 2 \) (since \( N \) is ample). Hence \( n_i \geq 0 \) for all \( i \) and \( n = \sum_{i=1}^{r} n_i \geq 0 \). Since \( D^2 = 0 \), in fact \( n > 0 \).

We consider two different cases: \( n \geq r \) or \( n < r \).

First, suppose that \( n \geq r \). Since \( f^2 = \sum_{i=1}^{r} n_i^2 \geq \frac{n^2}{r} \) and by hypothesis, \( d+3 > \frac{r}{e}(m+1) \), we have

\[
1 = D \cdot N = (d+3)f - n(m+1) > \frac{n \sqrt{t}}{e} (m+1) - n(m+1).
\]

We claim that \( \frac{n \sqrt{t}}{e} - n \geq 1/2 \) for any fixed \( e \), all \( r \geq e^2 + 1 \) and for all \( n \geq r \). Since this is a linear function in \( n \) of positive slope, it suffices to show that the function is non-negative when \( n = r \). That is, we only have to show that \( \frac{(\sqrt{t})r}{e} - r \geq 1/2 \) for \( r \geq e^2 + 1 \). For a fixed \( e \), this function is increasing for \( r > 0 \). So it suffices to show that \( \frac{\sqrt{e^2 + 1}}{e} - (e^2 + 1) \geq 1/2 \). It is easy to see that this inequality holds for \( e > 1 \), for example by clearing the denominator and squaring.

Thus, by (3.1), \( 1 > \left( \frac{n \sqrt{t}}{e} - n \right) (m+1) \geq \frac{m+1}{2} \geq 1 \), which is a contradiction.

Finally, we consider the case \( n < r \). We have \( f^2 = \sum_{i=1}^{r} n_i^2 \geq n \). So \( 1 = N \cdot D = (d+3)f - n(m+1) \geq \left( \frac{r \sqrt{t}}{e} - n \right) (m+1) \). We claim that \( \frac{r \sqrt{t}}{e} - n \geq 1 \), which as above leads to a contradiction.

We view \( \frac{r \sqrt{t}}{e} - n \) as a quadratic in \( \sqrt{t} \). Since the leading coefficient is \(-1\), it is a down-ward sloping parabola. If we show that the value of this function is at least \( 1 \) for \( n = 1 \) and \( n = r - 1 \), then it follows that the value of the function is at least \( 1 \) for all \( 1 \leq n \leq r - 1 \). This can be easily verified.

We conclude that \( L \) is globally generated. \( \square \)
Now we will consider the case \( k \geq 1 \). Recall the criterion \([1\text{, Theorem 2.1}]\) of Beltrametti-Francia-Sommese for \( k \)-very ampleness of \( L \), which generalizes Reider’s criterion in Theorem \([3.1]\)

**Theorem 3.3** (Beltrametti-Francia-Sommese). Let \( N \) be a line bundle on a surface \( X \). Let \( k \geq 0 \) be an integer. Suppose that \( N \) is nef and \( N^2 \geq 4k + 5 \). If \( K_X + N \) is not \( k \)-very ample, then there exists an effective divisor \( D \) on \( X \) such that

\[
N \cdot D - k - 1 \leq D^2 < \frac{N \cdot D}{2} < k + 1.
\]

If \((3.2)\) holds for an effective divisor \( D \), then our next two results give some conditions that \( D \) must satisfy.

**Proposition 3.4.** Let \( X \) be as in Theorem \([3.2]\). Let \( L = dH - m \sum_{i=1}^{r} E_i \) and \( N = L - K_X = (d + 3)H - (m + 1) \sum_{i=1}^{r} E_i \) be line bundles on \( X \). Let \( k \geq 1 \) be an integer. Suppose that \( r \geq e^2 + k + 1, (d + 3)e > r(m + 1) \) and \( m \geq k \). If an effective divisor \( D = fH - \sum_{i=1}^{r} n_i E_i \) on \( X \) satisfies \((3.2)\), then \( \sum_{i=1}^{r} n_i < r \).

**Proof.** Set \( n = \sum_{i=1}^{r} n_i \). Let \( \alpha = N \cdot D = (d + 3)f - (m + 1)n \) and \( \beta = D^2 = r^2 - \sum_{i=1}^{r} n_i^2 \). If \( D \) satisfies \((3.2)\), then we have

\[
(3.3) \quad \alpha - k - 1 \leq \beta < \frac{\alpha}{2} < k + 1.
\]

By hypothesis, \((d + 3)^2 > \frac{r^2}{e^2} (m + 1)^2 \). Since \((d + 3)f = \alpha + (m + 1)n \), we have

\[
(d + 3)^2 f^2 = (m + 1)^2 n^2 + 2(m + 1)n \alpha + \alpha^2 > \frac{r^2}{e^2} (m + 1)^2 \left( \beta + \sum_{i=1}^{r} n_i^2 \right).
\]

Since \( r \sum_{i=1}^{r} n_i^2 \geq n^2 \), we have

\[
(m + 1)^2 n^2 + 2(m + 1)n \alpha + \alpha^2 > \frac{r^2}{e^2} (m + 1)^2 \beta + \frac{r}{e^2} (m + 1)^2 n^2.
\]

We now show that the above inequality is impossible for \( n \geq r \). Specifically, we make the following claim.

**Claim:** Let \( r, e, k, \alpha, \beta \) be as in the proposition and suppose that \((3.3)\) holds. Then for \( n \geq r \), we have

\[
\frac{r^2}{e^2} (m + 1)^2 \beta + \frac{r}{e^2} (m + 1)^2 n^2 \geq (m + 1)^2 n^2 + 2(m + 1)n \alpha + \alpha^2.
\]

**Proof of Claim:** We consider the difference of the two terms in the required inequality as a quadratic function in \( n \). Define

\[
\lambda(n) := \left( \frac{r}{e^2} - 1 \right) (m + 1)^2 n^2 - 2(m + 1)n \alpha + \frac{r^2 \beta}{e^2} (m + 1)^2 - \alpha^2.
\]

This is quadratic in \( n \) with the leading coefficient \((\frac{r}{e^2} - 1)(m + 1)^2 > 0\). We will show that \( \lambda(r) \geq 0 \) and \( \lambda'(r) \geq 0 \), which will prove the claim and the proposition.
\[ \lambda(r) = r^2(m+1)^2 \left( \frac{r+\beta}{e^2} - 1 \right) - 2(m+1)\alpha r - \alpha^2 \]
\[ \geq r^2(m+1)^2 \left( \frac{e^2+k+1+\beta}{e^2} - 1 \right) - 2(m+1)\alpha r - \alpha^2 \quad \text{(since } r \geq e^2+k+1) \]
\[ = r^2(m+1)^2 \left( \frac{k+1+\beta}{e^2} \right) - 2(m+1)\alpha r - \alpha^2 \]
\[ \geq r(e^2+k+1)(m+1)^2 \left( \frac{k+1+\beta}{e^2} \right) - 2(m+1)\alpha r - \alpha^2 \]
\[ = r(m+1)^2(k+\beta+1) + \frac{r(k+1)(m+1)^2(k+\beta+1)}{e^2} - 2(m+1)\alpha r - \alpha^2 \]
\[ \geq r(m+1)^2(k+\beta+1) + (k+1)(m+1)^2(k+\beta+1) - 2(m+1)\alpha r - \alpha^2 \]
\[ \geq 0. \]

The last inequality follows when we compare the first term with the third term and the second term with the fourth term. We use the inequalities \( \alpha \leq \beta + k + 1 \) and \( \alpha < 2(k+1) \) which hold by (3.3), and \( m \geq k \geq 1 \), which holds by hypothesis.

Next we show that \( \lambda'(r) \geq 0 \).
\[ \lambda'(n) = 2 \left( \frac{r}{e^2} - 1 \right) (m+1)^2 n - 2(m+1)\alpha. \] Thus
\[ \lambda'(r) = \left( \frac{2r^2}{e^2} - 2r \right) (m+1)^2 - 2(m+1)\alpha \]
\[ \geq \frac{2r(e^2+k+1)}{e^2} (m+1)^2 - 2r(m+1)^2 - 2(m+1)\alpha \]
\[ = 2r(m+1)^2 + 2r(m+1)^2 \frac{k+1}{e^2} - 2r(m+1)^2 - 2(m+1)\alpha \]
\[ = 2r(m+1)^2 \frac{k+1}{e^2} - 2(m+1)\alpha \]
\[ \geq 2(m+1)^2(k+1) - 2(m+1)\alpha \]
\[ \geq 0 \quad \text{(by (3.3) and the hypothesis that } m \geq k \geq 1). \]

This completes the proof of the proposition. \( \square \)

**Proposition 3.5.** Let \( L = dH - m \sum_{i=1}^r E_i \) and \( N = L - K_X = (d+3)H - (m+1) \sum_{i=1}^r E_i \).

Let \( k \) be a positive integer. Suppose that \( r \geq e^2+k+1 \), \( (d+3)e > r(m+1) \) and \( m \geq k \). Let \( D \) be an effective divisor on \( X \) such that \( D \cdot C_1 \geq 0 \). Then \( D \) does not satisfy (3.2).

**Proof.** Let \( D = fH - \sum_{i=1}^r n_i E_i \). Then \( f \geq 0 \) and if \( f = 0 \), then \( n_i \leq 0 \) for all \( i = 1, \ldots, r \).

Let \( n = \sum_{i=1}^r n_i \). Then \( D^2 = f^2 - \sum_{i=1}^r n_i^2 \leq f^2 - n \). Indeed, this follows because \( \sum_{i=1}^r n_i^2 \geq n \). Moreover, the assumption \( D \cdot C_1 \geq 0 \) implies that \( ef \geq n \).

Suppose that \( D \) satisfies (3.2). We will obtain a contradiction.
First let \( f = 0 \). We have \( N \cdot D = -(m+1)n < 2(k+1) \leq 2(m+1) \). So \( n > -2 \). On the other hand, \( 0 = ef \geq n \). So \( n = 0 \) or \( n = -1 \). Since each \( n_i \) is non-positive, if \( n = 0 \) then \( D = 0 \). If \( n = -1 \), then \( D = E_i \) for some \( i \). But then \( N \cdot D = m+1 \geq k+1 \), hence \( N \cdot D - k - 1 \geq 0 \), while \( D^2 = -1 \). This violates (3.2).

Let \( f = 1 \). In this case, we have \( e \geq n \). We may also assume \( n > 0 \). Then \( 2(k+2) > N \cdot D = (d+3) - n(m+1) > (r/n)(m+1) \geq (e-n)(m+1) + (k+1)(m+1) \geq (e-n)(k+1) + (k+1/e)(k+1) \). Thus \( 0 \leq e - n < 2 \). So \( e = n \) or \( e = n+1 \).

Let \( e = n+1 \). Then \( 0 \geq 1-n \geq D^2 \geq N \cdot D - k - 1 \geq k+1 + (k+1/e)(k+1) - k - 1 > 0 \), which is a contradiction. If \( e = n \), then \( 0 \geq 1-n \geq D^2 \geq N \cdot D - k - 1 \geq (k+1)(k+1) - k - 1 \). Hence \( k+1 < e \). But then \( D^2 \leq 1-n = 1-e < -k \leq N \cdot D - k - 1 \). The last inequality holds because \( N \) is ample and hence \( N \cdot D > 0 \). Again we have a contradiction, because this violates (3.2).

Now suppose that \( f \geq 2 \). As above, we have \( 2(k+2) > N \cdot D > r(m+1)/e - ef(m+1) = (r/e-ef)(m+1) \geq (k+1/e)(m+1) \).

Thus \( 2 > (m+1)/e \), or equivalently, \( f < 2e/m+1 \).

If \( e \leq k+1 \), then \( f < 2e/m+1 \leq 2e/k+1 \leq 2 \), which contradicts the hypothesis \( f \geq 2 \). So we may assume \( e > k+1 \).

We now make the following claim:

**Claim:** \( f^2 - n \leq 2 - 2k \).

**Proof of Claim:** Note that \( ef - n \leq 1 \). Indeed, \( (d+3)f > r/f(m+1) > ef(m+1) \). Hence \( 2(k+1) > N \cdot D = (d+3)f - n(m+1) > (e-n)(m+1) \geq (ef-n)(k+1) \). Thus \( ef-n < 2 \). On the other hand, by the hypothesis in the proposition \( ef-n \geq 0 \). Hence we have:

Thus \( ef-n = 0 \) or \( ef-n = 1 \).

On the other hand,

\[
f^2 - n < f \left( \frac{2e}{m+1} \right) - n \quad \text{(by (3.4))}
= ef + \frac{ef(1-m)}{m+1} - n
\leq \frac{ef(1-k)}{k+1} + ef - n \quad \text{(since } m \geq k)^
\leq \frac{ef(1-k)}{k+1} + 1 \quad \text{(by (3.5))}
\leq f(1-k) + 1 \quad \text{(since } e \geq k+1 \text{ and } 1-k \leq 0)^
\leq 2(1-k) + 1 = 3 - 2k \quad \text{(since } f \geq 2 \text{ and } 1-k \leq 0)^
\]
This completes the proof of the claim. Now we consider three cases:

\( k \geq 3 \): In this case, \( f^2 - n \leq 2 - 2k \leq -k - 1 \). So \( D^2 = f^2 - \sum_{i=1}^{r} n_i^2 \leq f^2 - n \leq -k - 1 \). But by (3.2), we have \( N \cdot D - k - 1 \leq D^2 \). Since \( N \) is ample, \( -k \leq N \cdot D - k - 1 \), contradicting the inequality \( D^2 \leq -k - 1 \).

\( k = 1 \): Then by (3.2), the claim, and the fact that \( N \cdot D \geq 1 \), we have \(-1 \leq N \cdot D - 2 \leq f^2 - n \leq 0 \). By (3.5), we have either \( ef = n \) or \( ef - 1 = n \).

If \( ef = n \), then \( 0 \geq f^2 - n = f(f - e) \geq -1 \). Since \( f \geq 2 \), the only possibility \( f = e \). But this violates (3.4), because \( m \geq k = 1 \). On other hand, if \( ef - 1 = n \), then \( 0 \geq f^2 - n = f^2 - ef + 1 = f(f - e) + 1 \geq -1 \). Since \( f \geq 2 \), \( f^2 - n \) must be -1. But then the only possibility is \( f = e = 1 \) and again we have a contradiction to (3.4).

\( k = 2 \): By (3.2), the claim, and the fact that \( N \cdot D \geq 1 \), we have \(-2 \leq N \cdot D - 3 \leq f^2 - n \leq -2 \). Hence \( f^2 - n = -2 \). If \( ef = n \), then \(-2 = f^2 - ef = f(f - e) \). This contradicts (3.4). If \( ef - 1 = n \), then \(-2 = f(f - e) + 1 \). Again we obtain a contradiction to (3.4).

This completes the proof of the claim. \( \square \)

Now we are ready to prove our main result on \( k \)-very ampleness.

**Theorem 3.6.** Let \( C \) be an irreducible and reduced plane curve of degree \( e \). Let \( X \to \mathbb{P}^2 \) be the blow up of \( \mathbb{P}^2 \) at \( r \) distinct smooth points \( p_1, \ldots, p_r \in C \). Let \( H \) denote the pull-back of \( \mathcal{O}_{\mathbb{P}^2}(1) \) and let \( E_1, \ldots, E_r \) be the inverse images of \( p_1, \ldots, p_r \) respectively. Let \( k \) be a non-negative integer. Let \( L = dH - m \sum_{i=1}^{r} E_i \) be a line bundle on \( X \) with \( m \geq k \).

If \((d + 3)e > r(m + 1)\) and \( r \geq e^2 + k + 1 \), then \( L \) is \( k \)-very ample.

**Proof.** When \( k = 0 \), this is the same as Theorem 3.2. So we will assume that \( k \geq 1 \) and use the criterion of Beltrametti-Francia-Sommese.

Let \( N = L - K = (d + 3)H - (m + 1) \sum_{i=1}^{r} E_i \). Just as in the proof of Theorem 3.2, we conclude that \( N \) is ample.

Next we claim that \( N^2 \geq 4k + 5 \). Indeed, we have

\[
N^2 = (d + 3)^2 - r(m + 1)^2 \\
> \frac{r^2}{e^2}(m + 1)^2 - r(m + 1)^2 \\
= (m + 1)^2 r \left( \frac{r}{e^2} - 1 \right) \\
\geq (m + 1)^2 r \left( \frac{k + 1}{e^2} \right) \\
> (m + 1)^2 (k + 1) \\
\geq 4(k + 1) \quad ( \text{since } m \geq k \geq 1 ).
\]
Since $N^2$ is an integer and $N^2 > 4k + 4$, we conclude that $N^2 \geq 4k + 5$. Hence we can apply the criterion of Beltrametti-Francia-Sommese. Suppose that $L$ is not $k$-very ample. Then there exists an effective divisor $D$ on $X$ such that (3.2) holds. We will obtain a contradiction.

Write $D = fH - n_1E_1 - \ldots - n_rE_r$ and let $n = \sum_i n_i$. By Proposition 3.4 we have $n < r$. By Proposition 3.5 $D \cdot C_1 < 0$. This implies that $C_1$ is a component of $D$ and we may write $D = aC_1 + D'$ for a positive integer $a$ and an effective divisor $D'$.

Write $D' = bH - \sum_{i=1}^r l_iE_i$ with $b \geq 0$. Then we have $f = ae + b$. So

$$2(k+1) > N \cdot D = (d+3)f - (m+1)n > \frac{r}{e}(ae + b)(m+1) - (r-1)(m+1) \quad \text{(since } n \leq r - 1)$$

$$= (r(a-1) + 1)(m+1) + \frac{rb}{e}(k+1).$$

Thus $a = 1$ and $b = 0$. In particular, $f = e$. We have $2(k+1) > N \cdot D = (d+3)e - n(m+1) > (r-n)(m+1)$. Hence $r-n < 2$. On the other hand, $r-n \geq 1$. So $r-n = 1$.

Since $N$ is ample, $N \cdot D > 0$. Thus $-k \leq N \cdot D - k - 1 \leq D^2 = e^2 - \sum_{i=1}^r n_i^2 \leq e^2 - n = e^2 - r + 1 \leq -k$. The last inequality holds because $r \geq e^2 + k + 1$. Thus we have $D^2 = -k$ and $N \cdot D = 1$. But $N \cdot D > m + 1 \geq k + 1$. This is a contradiction.

The proof of the theorem is complete. \hfill \Box

**Remark 3.7.** [7] Theorem 4.1] gives conditions for $k$-very ampleness for any line bundle on the blow up of $\mathbb{P}^2$ at general points on an irreducible and reduced cubic. In our context, this is the case $e = 3$. If the line bundle is uniform, that is if $L = dH - m \sum_{i=1}^r E_i$ and if the number of points $r$ is at least $10+k$, then our Theorem 3.6 is comparable to this result. However, we note that [7] Theorem 4.1] deals with any (not just uniform) line bundle $L = dH - \sum_{i=1}^r m_iE_i$ and any $r \geq 3$.

The next two examples show that the hypothesis of Theorem 3.6 can not be weakened for $e = 3$.

**Example 3.8.** Let $C$ be a smooth plane cubic. Let $X$ be the blow up of 10 distinct points on $C$. Consider the line bundle $L = 7H - 2 \sum_{i=1}^{10} E_i$ on $X$. We have $C_1 = 3H - \sum_{i=1}^{10} E_i$. By Corollary 2.4 $L$ is ample. We use [2, Corollary 1.4] to show that $L$ is not globally generated. According to this result, if a line bundle on a curve of positive genus is $k$-very ample, then the degree of the line bundle is at least $k+2$. In our example, if $L$ is globally generated (that is, if it is 0-very ample), then $L|_C$ is also 0-very ample on $C$. But deg($L|_C$) = $L \cdot C = 1 < 2$. So the strict inequality, $e(d + 3) > r(m + 1)$, in the hypothesis of Theorem 3.6 can not be relaxed.

**Example 3.9.** This is a small variation on Example 3.8. Again let $e = 3$, $r = 10$, but now let $m = 7$. Consider $L = 24H - 7 \sum_{i=1}^{10} E_i$. It is easy to check that $L$ is ample (by Corollary 2.4)
and globally generated (by Theorem 3.6). But \( L \) is not very ample because \( L \cdot C_1 = 2 < 1 + 2 \) (again by [2, Corollary 1.4]). Here the hypothesis in Theorem 3.6 on the number of points (namely, \( r \geq e^2 + k + 1 = 10 + k \)) does not hold.

**Example 3.10.** If \( e > 3 \), our hypotheses in Theorem 3.6 are not likely to be optimal. We will illustrate this with just one example.

Let \( e = 5, k = 5, r = 31 \) and consider \( L_d = dH - 5(E_1 + \ldots + E_{31}) \). By Theorem 3.6 \( L_{35} \) is 5-very ample. On the other hand, [2, Corollary 1.4] shows that \( L_{32} \) is not 5-very ample. We do not know if \( L_{33} \) or \( L_{34} \) is 5-very ample. Note that the criterion of Beltrametti-Francia-Sommese (Theorem 3.3) can not be applied here, since \( N_d = L_d - K_X \) is not nef for \( d < 35 \). Indeed, \( N_d \cdot C_1 = 5(d + 3) - 186 < 0 \), for \( d < 35 \). In other words, our method (which is to use Theorem 3.3 to show \( k \)-very ampleness) itself is not applicable to \( L_{33} \) and \( L_{34} \).

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