POSITIVITY OF VECTOR BUNDLES ON HOMOGENEOUS VARIETIES

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Abstract. We study the following question: Given a vector bundle on a projective variety X such that the restriction of E to every closed curve $C \subset X$ is ample, under what conditions E is ample? We first consider the case of an abelian variety X. If E is a line bundle on X, then we answer the question in the affirmative. When E is of higher rank, we show that the answer is affirmative under some conditions on E. We then study the case of $X = G/P$, where G is a reductive complex affine algebraic group, and P is a parabolic subgroup of G. In this case, we show that the answer to our question is affirmative if E is T–equivariant, where $T \subset P$ is a fixed maximal torus. Finally, we compute the Seshadri constant for such vector bundles defined on G/P.

Contents

1. Introduction

Let X be a projective variety defined over an algebraically closed field, and let L be a line bundle on X. The Nakai–Moishezon criterion says that L is ample if and only if $L^{\dim(Y)} \cdot Y > 0$ for every positive-dimensional closed subvariety Y of X. In general, it is not sufficient to check this condition only for the closed curves in X. Mumford gave an example of a non-ample line bundle on a surface which intersects every closed curve positively; see [Har2, Example 10.6] or [La1, Example 1.5.2].

However, in some cases it turns out that to check ampleness of L it suffices to verify the condition $L \cdot C > 0$ for all closed curves $C \subset X$. Line bundles satisfying this condition are called *strictly nef*; see [Se]. Strictly nef divisors have been studied by many authors.

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Strictly nef divisors have interesting connections to many questions; for more details, see [CCP, Se].

Mumford's example gives a strictly nef, but not ample, divisor on a ruled surface. One can still ask the following question:

Under what situations is a strictly nef divisor ample?

Examples where the answer is known to be positive are provided by abelian varieties and toric varieties.

In this note, we ask the following:

Given a vector bundle E on a projective variety X such that the restriction of E to every closed curve $C \subset X$ is ample, under what conditions is E ample?

In [HMP], this question is studied for toric varieties; in fact, in [HMP] it is proved that an equivariant vector bundle E on a toric variety X is ample if the restriction of E to the invariant rational curves on X is ample. We recall that there are only finitely many invariant rational curves on X. For a flag variety X over a projective curve defined over $\overline{\mathbb{F}}_p$, a line bundle on X is ample if its restriction to each closed curve is ample [BMP]. When X is a wonderful compactification, this question is studied in [BKN].

Here we address the above question for abelian varieties and homogeneous varieties G/P, where G is a reductive affine algebraic group over $\mathbb{C}$, and P is a parabolic subgroup of G.

In Section 2, we consider abelian varieties. The case of line bundles on abelian varieties is known, but we start with by giving a proof in this case, for completeness (Proposition 2.1). We then consider vector bundles on abelian varieties and answer the question in the affirmative under some conditions (see Proposition 2.3). Our result shows, in particular, that a homogeneous vector bundle on an abelian variety has the above mentioned property.

In Section 3, we consider homogeneous varieties G/P, where G and P are as above. Let T be a maximal torus of G contained in P. We show that a T–equivariant vector bundle has positive answer to our question (see Theorem 3.1). Finally, we calculate Seshadri constants for T–equivariant bundles on G/P at T–fixed points (see Theorem 3.3).

2. Vector bundles on an abelian variety

Let k be an algebraically closed field. We first consider the case of line bundles on abelian varieties. In this case it is known that our question has a positive answer; see [Se, Proposition 1.4] for example[1]. We still include a proof below for the sake of completeness.

Proposition 2.1. *Let A be an abelian variety defined over k. Let L be a line bundle over A satisfying the following condition: for every pair (C, f), where C is an irreducible smooth projective curve defined over k, and $f : C \longrightarrow A$ is a non-constant morphism, the inequality*

$$\text{degree}(f^*L) > 0 \tag{2.1}$$

[1]We thank Patrick Brosnan for pointing out this reference to us.

holds. Then L is ample.

Proof. Take a line bundle $\mathcal{L}$ on A. Let

$$\alpha : A \times A \longrightarrow A, \quad (x, y) \longmapsto x + y$$

be the addition map. Consider the family of line bundles

$$(\alpha^*\mathcal{L}) \otimes p_1^*\mathcal{L}^* \longrightarrow A \times A \xrightarrow{p_2} A,$$

where p_1 and p_2 are the projections of $A \times A$ to the first and second factor respectively. Let

$$\varphi_{\mathcal{L}} : A \longrightarrow A^\vee = \mathrm{Pic}^0(A)$$

be the classifying morphism for this family. This $\varphi_{\mathcal{L}}$ is a group homomorphism. Let

$$K(\mathcal{L}) \subset A$$

be the (unique) maximal connected subgroup of the reduced kernel $\ker(\varphi_{\mathcal{L}})_{\mathrm{red}}$.

If $\mathcal{L} \in A^\vee = \mathrm{Pic}^0(A)$, then $\varphi_{\mathcal{L}}$ is the constant morphism $x \longmapsto 0$ [MFK, p. 120] (see after Definition 6.2), [GN, p. 11, Lemma 2.1.6]. Using this it follows that if $\mathcal{L}'$ is numerically equivalent to $\mathcal{L}$, then $\varphi_{\mathcal{L}} = \varphi_{\mathcal{L}'}$, which in turn implies that

$$K(\mathcal{L}) = K(\mathcal{L}'). \tag{2.2}$$

It is known that $\mathcal{L}$ is ample if the following two conditions hold:

(1) the line bundle $\mathcal{L}$ is effective, and
(2) $K(\mathcal{L}) = 0$.

(See [Mum1, p. 288, § 1], [GN, p. 13, Theorem 2.2.2].)

We will use the following lemma:

Lemma 2.2. *The line bundle L in Proposition 2.1 is ample if $K(L) = 0$.*

Proof of Lemma 2.2. Since L is nef, it follows that L is numerically equivalent to a $\mathbb{Q}$–effective $\mathbb{Q}$–Cartier divisor on A (see [Mo, p. 811, Proposition 3.1]). So $L^{\otimes n}$ is numerically equivalent to an effective divisor D on A, for some positive integer n. Note that

$$\varphi_{L^n} = n \cdot \varphi_L . \tag{2.3}$$

Assume that $K(L) = 0$. Consequently, from (2.3) and (2.2) it follows that $K(D) = 0$. Since D is also effective, using the above mentioned criterion for ampleness it follows that D is ample. This implies that L is ample. □

Continuing with the proof of Proposition 2.1, in view of Lemma 2.2 it suffices to show that $\dim K(L) = 0$. Assume that

$$\dim K(L) \geq 1 .$$

The restriction of L to the sub-abelian variety $K(L) \subset A$ will be denoted by L_0. For any closed point $x \in A$, define

$$\alpha_x : A \longrightarrow A, \quad y \longmapsto x + y . \tag{2.4}$$

For any closed point $x \in K(L)$, let $\widehat{\alpha}_x : K(L) \longrightarrow K(L)$ be the restriction of α_x in (2.4) to $K(L)$.

For any $x \in K(L)$, we have $\alpha_x^* L = L$; hence we have

$$\widehat{\alpha}_x^* L_0 = (\alpha_x^* L)|_{K(L)} = L|_{K(L)} = L_0 .$$

This implies that the line bundle L_0 on $K(L)$ is numerically trivial [Mum2, p. 74, Definition] and [Mum2, p. 86]. Consequently, for any pair (C, f), where C is an irreducible smooth projective curve defined over k, and $f : C \longrightarrow K(L) \subset A$ is a non-constant morphism, we have

$$\text{degree}(f^* L) = 0 .$$

Since this contradicts (2.1), we conclude that $\dim K(L) = 0$. As observed above, this implies that L is ample. □

2.1. **Ample vector bundles on A.** As before A is an abelian variety. Let E be a vector bundle of rank r over A satisfying the following two conditions:

(1) The line bundle $\det E := \bigwedge^r E$ has the property that for every pair (C, f), where C is an irreducible smooth projective curve defined over k, and $f : C \longrightarrow A$ is a non-constant morphism, the inequality $\text{degree}(f^* \det E) > 0$ holds.
(2) for every closed point $x \in A$, there is a line bundle $L(x)$ on A such that

$$\alpha_x^* E = E \otimes L(x) , \tag{2.5}$$

where α_x is the morphism in (2.4).

Proposition 2.3. *The above vector bundle E on A is ample.*

Proof. Since E satisfies the condition in (2.5), a theorem of Mukai says that there is an isogeny

$$f : B \longrightarrow A$$

such that the vector bundle $f^* E$ admits a filtration of subbundles

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E \tag{2.6}$$

for which $\text{rank}(E_i) = i$, and the line bundle E_i/E_{i-1} is numerically equivalent to E_1 for every $1 \leq i \leq r$ [Muk, p. 260, Theorem 5.8] (see also [MN, p. 2]).

Consequently, the line bundle $\det E = \bigwedge^r E = \bigotimes_{i=1}^r (E_i/E_{i-1})$ is numerically equivalent to the line bundle $E_1^{\otimes r}$. From Proposition 2.1 we know that $\det E$ is ample. This implies that $E_1^{\otimes r}$ is ample. Hence E_1 is ample. So E_i/E_{i-1} is ample for every $1 \leq i \leq r$. Consequently, from (2.6) it follows that E is ample [La2, p. 13, Proposition 6.1.13]. □

Let X be a projective variety defined over an algebraically closed field k. A divisor D on X is said to be *big* if there is an ample divisor H on X such that the difference $mD - H$ is linearly equivalent to an effective divisor for some positive integer m. A $\mathbb{Q}$–divisor D is *pseudo-effective* if $D + B$ is big for any big $\mathbb{Q}$–divisor B. Similarly one can define the notion of pseudo-effective $\mathbb{R}$–divisors. In the Néron–Severi space $N^1(X)_{\mathbb{R}}$, the pseudo-effective $\mathbb{R}$–divisors form a cone which is the closure of the cone of effective $\mathbb{R}$–divisors.

If $\dim X = 2$, and the pseudo-effective cone of X is equal to the cone of effective divisors, then a line bundle L on X is ample if and only if $L \cdot C > 0$ for every closed curve C on X. But, in general, the pseudo-effective cone of a projective variety is not equal to the effective cone; see the example of Mumford described in [Har2, Example 10.6] or [La1, Example 1.5.2].

If k is an algebraic closure of a finite field, Moriwaki showed that every pseudo-effective divisor (over $\mathbb{Q}$ or $\mathbb{R}$) is effective when X is a projective bundle over a projective curve or when X is an abelian variety (see [Mo, p. 802, Theorem 0.4] and [Mo, p. 802, Proposition 0.5]). As our next example shows, this statement is false for abelian varieties over $\mathbb{C}$.

Example 2.4. Let X be an elliptic curve defined over $\mathbb{C}$. Let $x \in X$ be a point of infinite order. Let $D := 0 - x$, where 0 is the identity element of X. Then D is a divisor of degree 0 and it is pseudo-effective. However, no multiple of D is effective, since x has infinite order.

In view of Proposition 2.3, it is natural to ask the following:

Question 2.5. Let A be an abelian variety over an algebraically closed field. Let E be a vector bundle on A such that the restriction $E|_C$ is ample for every closed curve C on A. Is this E ample?

3. Equivariant vector bundles on G/P

Theorem 3.1. *Let G be a reductive affine algebraic group defined over $\mathbb{C}$, and let $P \subset G$ be a parabolic subgroup. Fix a maximal torus $T \subset G$ such that $T \subset P$. Let E be a T–equivariant vector bundle on G/P. Then E is nef (respectively, ample) if and only if the restriction $E|_C$ of E to every T–invariant closed curve C on G/P is nef (respectively, ample).*

Proof. Let $Y(T)$ be the group of all 1–parameter subgroups of T. Note that $Y(T)$ is a finitely generated free abelian group whose rank is equal to the dimension of T. Let

$$\{\lambda_1, \cdots, \lambda_n\} \tag{3.1}$$

be a basis of the $\mathbb{Z}$–module $Y(T)$.

If E is nef (respectively, ample), then clearly $E|_C$ is nef (respectively, ample) for every closed curve C on G/P.

To prove the converse, first assume that E is a T–equivariant vector bundle on G/P such that its restriction $E|_C$ to every T–invariant curve $C \subset G/P$ is nef.

Let

$$\pi : \mathbb{P}(E) \longrightarrow G/P$$

be the projective bundle over G/P parametrizing the hyperplanes in the fibers of E. The tautological relative ample line bundle over $\mathbb{P}(E)$ will be denoted by $\mathcal{O}_{\mathbb{P}(E)}(1)$. To prove that E is nef, we need to show that $\mathcal{O}_{\mathbb{P}(E)}(1)|_D$ is nef for every closed curve $D \subset \mathbb{P}(E)$. Note that if $\pi(D)$ is a point, then $\mathcal{O}_{\mathbb{P}(E)}(1)|_D$ is ample, because $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively ample.

Therefore, we can assume that $\pi(D)$ is a closed curve in G/P. Let $\widetilde{D_1}$ be the flat limit of the curves $\lambda_1(t)D$ (see (3.1)) as $t \to 0$. In other words, $\widetilde{D_1}$ is a 1–cycle which corresponds to the limit of the points $\lambda_1(t)[D]$ (as $t \to 0$) in the Hilbert scheme of curves in $\mathbb{P}(E)$. Note that since E is T–equivariant, the 1–parameter subgroup λ_1 acts on the Hilbert scheme of curves in $\mathbb{P}(E)$. It follows that the 1–cycle $\widetilde{D_1}$ and $D_1 := \pi(\widetilde{D_1})$ are λ_1–invariant. Now let $\widetilde{D_2}$ be the flat limit of $\lambda_2(t)\widetilde{D_1}$ as $t \to 0$. Since λ_1 and λ_2 commute, we see that $\widetilde{D_2}$ and $\pi(\widetilde{D_2})$ are invariant under both λ_1 and λ_2. Continuing this way, we obtain a 1–cycle $\widetilde{D_n}$ on $\mathbb{P}(E)$ such that both $\widetilde{D_n}$ and $\pi(\widetilde{D_n})$ are invariant under $\lambda_1, \cdots, \lambda_n$. Consequently, both $\widetilde{D_n}$ and $\pi(\widetilde{D_n})$ are T–invariant.

Now, by the assumption on E, we have

$$\text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\widetilde{D_n}}) \geq 0\,. \tag{3.2}$$

Since the curve D is linearly equivalent to $\widetilde{D_n}$, from (3.2) it follows immediately that $\text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)|_D) \geq 0$. This proves that E is nef if $E|_C$ is nef for every closed T–invariant curve $C \subset G/P$.

Next we shall prove that E is ample if $E|_C$ is ample for every T–invariant closed curve $C \subset G/P$.

Note that every line bundle on G/P is T–equivariant. Hence if F is a T–equivariant vector bundle on G/P, then so is $F \otimes L$ for any line bundle L on G/P.

We claim that the set of T–invariant closed curves in G/P is finite.

To prove the above claim, let B be a Borel subgroup of G containing T and contained in P. Then B acts on G via left–translations. Let $W := N_G(T)/T$ be the corresponding Weyl group, and let $W_P \subset W$ be the subgroup consisting of elements that preserve P. Note that G is the disjoint union of double cosets BwP, where w runs through W/W_P. This gives the Bruhat decomposition

$$G/P = \cup_{w \in W/W_P} BwP$$

(see [Hu, § 29.2], [Hu, § 29.3]). It is clear that the T–fixed points in G/P are precisely wP for every $w \in W/W_P$. Further, in each of the B–orbits BwP in G/P, there are only finitely many T–invariant curves, namely the images of the subgroups generated by the root vectors in B. Since W is finite, we conclude that G/P has only finitely many T–fixed points and T–invariant curves. This proves the claim.

The ampleness of E now follows by a standard argument, which we include for the convenience of the reader. Fix an ample line bundle L on G/P. Let $\text{Sym}^n(E)$ denote the n–th symmetric power of E. Then for n sufficiently large, we have that $\text{Sym}^n(E) \otimes L^{-1}|_C$ is nef for every T–invariant curve C. By the first part of the theorem, the vector bundle $\text{Sym}^n(E) \otimes L^{-1}$ is nef. Since L is ample, this implies that $\text{Sym}^n(E) = (\text{Sym}^n(E) \otimes L^{-1}) \otimes L$ is ample, and consequently E itself is ample (see [La2, Proposition 6.2.11] and [Har1, p. 67, Proposition 2.4]). □

Proposition 3.2. *Let G, P and T be as in Theorem 3.1. Let $x \in G/P$ be a T–fixed point, and let $\varpi : \widetilde{X} \longrightarrow G/P$ denote the blow-up of G/P at x. Then the following three statements hold:*

(1) *The action of T lifts to $\widetilde{X}$.*
(2) *Let F be a T–equivariant vector bundle on $\widetilde{X}$. Then F is nef if and only if the restriction $F|_{\widetilde{C}}$ of F to every T–invariant closed curve $\widetilde{C} \subset X$ is nef.*
(3) *Let W_x denote the the exceptional divisor of the blow-up ϖ. Let E be a T–equivariant vector bundle on G/P. Then $(\varpi^\star E) \otimes \mathcal{O}_{\widetilde{X}}(W_x)^m$ is a T–equivariant vector bundle on $\widetilde{X}$ for every integer m.*

Proof. Since x is a T–fixed point, the group T acts on the tangent space $T_x(G/P) = \mathfrak{g}/\mathfrak{p}$, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{p}$ is the Lie algebra of P. Then the T–action on $T_x(G/P)$ decomposes it into a direct sum of one-dimensional T–invariant subspaces.

Note that the exceptional divisor of the blow-up $\varpi : \widetilde{X} \longrightarrow G/P$ is isomorphic to $\mathbb{P}(T_x(G/P))$. So T acts on the exceptional divisor of the blow-up via the linear action of T on $T_x(G/P)$. Since ϖ is an isomorphism outside the exceptional divisor, we conclude that the action of T lifts to all of $\widetilde{X}$. This proves (1).

Since the action of T lifts to $\widetilde{X}$, the proof of (2) goes through exactly as the proof of the analogous statement in Theorem 3.1. Note that in the proof of Theorem 3.1 we only used the T–action on G/P.

Now we prove (3). Since T acts on the exceptional divisor W_x of ϖ, we conclude that $\mathcal{O}_{\widetilde{X}}(W_x)^m$ is a T–equivariant line bundle for every integer m. Hence if E is a T–equivariant vector bundle on G/P then $(\varpi^\star E) \otimes \mathcal{O}_{\widetilde{X}}(W_x)^m$ is a T–equivariant vector bundle on $\widetilde{X}$ for every integer m. □

Let E be a vector bundle on a projective variety X. Take any point $x \in X$. The Seshadri constant of E at x was defined in [Hac]. This definition is recalled below.

Let $\varpi : \widetilde{X} \longrightarrow X$ denote the blow-up of X at x. Consider the following diagram, where p and q are projective bundles:

$$\begin{array}{ccc} \mathbb{P}(\varpi^\star E) & \xrightarrow{\widetilde{\varpi}} & \mathbb{P}(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \widetilde{X} & \xrightarrow{\varpi} & X \end{array} \tag{3.3}$$

Let

$$\xi = \mathcal{O}_{\mathbb{P}(\varpi^\star E)}(1) \tag{3.4}$$

be the tautological bundle on $\mathbb{P}(\varpi^\star E)$. Let $Y_x = p^{-1}(x)$ and $Z_x = \widetilde{\varpi}^{-1}(Y_x)$.

The Seshadri constant of E at x is defined as follows:

$$\varepsilon(E, x) := \sup\{\lambda \in \mathbb{Q}_{>0} \mid \xi - \lambda Z_x \text{ is nef}\}.$$

Here $\mathbb{Q}_{>0}$ denotes the set of positive rational numbers. For more details on Seshadri constants of vector bundles, see [Hac].

Now we work with the notation in Theorem 3.1. Let E be a T–equivariant vector bundle on G/P. It is known that each T–invariant closed curve $C \subset G/P$ is smooth rational. Indeed, let C be a T–invariant closed curve. Since there are only finitely many T–fixed points in G/P, there must exist a point $x \in C$ which is not fixed by T. Now consider the morphism $T \longrightarrow C$ which sends $t \in T$ to $t \cdot x \in C$. This is a non-constant morphism from a torus to C, and hence C must be rational. In the special case when $G = \mathrm{GL}_n(\mathbb{C})$, and P is the Borel subgroup of upper triangular matrices in G, a different proof for this fact can be found in [Bri, Page 44, 1.3.4, Example 2].

Let $C \subset G/P$ be a T–invariant closed curve. From a theorem of Grothendieck we know that the restriction of E to C has the form

$$E|_C \,=\, \mathcal{O}_C(a_1) \oplus \ldots \oplus \mathcal{O}_C(a_n)$$

for some integers $a_1, \cdots, a_n$ [Gr, p. 122, Théorème 1.1].

Theorem 3.3. *Let G be a complex reductive group, and let P be a parabolic subgroup of G containing a maximal torus T. Let E be a T–equivariant nef vector bundle on $X = G/P$ of rank n, and let $x \in X$ be a T–fixed point. Then*

$$\varepsilon(E, x) \,=\, min\{a_i(C)\}_{C,i}\,,$$

where the minimum is taken over all T–invariant curves $C \subset G/P$ passing through x and integers $\{a_1(C), \cdots, a_n(C)\}$ such that $E|_C = \mathcal{O}_C(a_1(C)) \oplus \cdots \oplus \mathcal{O}_C(a_n(C))$.

Proof. Let W_x denote the exceptional divisor of the blow-up

$$\varpi \,:\, \widetilde{X} \longrightarrow X \,=\, G/P$$

at the point $x \in G/P$. By definition, the Seshadri constant of E at x is given by the following:

$$\varepsilon(E, x) \,=\, \sup\{\lambda \in \mathbb{Q}_{>0} \ \mid\ \xi - \lambda q^\star(W_x) \ \text{ is nef}\}\,,$$

where q is the map in (3.3), ξ is the line bundle in (3.4) and W_x is the exceptional divisor as in the proof of Proposition 3.2(3).

We claim that $\xi - \lambda q^\star(W_x)$ is nef if $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x|_{\widetilde{C}})$ is nef for every T–invariant closed curve $\widetilde{C} \subset \widetilde{X}$. See the following diagram:

$$\begin{array}{ccccc}
\mathbb{P}(\varpi^\star E|_{\widetilde{C}}) & \longrightarrow & \mathbb{P}(\varpi^\star E) & \xrightarrow{\ \widetilde{\varpi}\ } & \mathbb{P}(E) \\
\downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
\widetilde{C} & \hookrightarrow & \widetilde{X} & \xrightarrow{\ \varpi\ } & X = G/P
\end{array}$$

The above claim essentially follows from the proof of Theorem 3.1 and Proposition 3.2(2). Indeed, to prove that $\xi - \lambda q^\star(W_x)$ is nef, we need to show that $(\xi - \lambda q^\star(W_x)) \cdot D \geq 0$ for every closed curve $D \subset \mathbb{P}(\varpi^\star E)$. But, as argued in the proof of Theorem 3.1, there exists a T–invariant curve $\widetilde{D} \subset \mathbb{P}(\varpi^\star E)$ which is linearly equivalent to D and such that $\widetilde{C} := q(\widetilde{D}) \subset \widetilde{X}$ is a T–invariant curve. But we have

$$(\xi - \lambda q^\star(W_x)) \cdot \widetilde{D} \,=\, \text{degree}(E|_{\widetilde{C}}) - \lambda(W_x \cdot \widetilde{C}) \,\geq\, 0\,.$$

The last inequality follows from the hypothesis that $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x|_{\widetilde{C}})$ is nef for every invariant curve $\widetilde{C} \subset \widetilde{X}$. This proves the claim.

Now let $\widetilde{C} \subset \widetilde{X}$ be any T–invariant curve. We know that $\widetilde{C}$ is isomorphic to the projective line $\mathbb{P}^1$. We will show below that $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x|_{\widetilde{C}})$ is nef.

First suppose that $\widetilde{C}$ is contained in the exceptional divisor W_x of the blow-up $\varpi : \widetilde{X} \longrightarrow X$. Note that W_X is isomorphic to a projective space, and

$$\mathcal{O}_{W_x}(W_x) = \mathcal{O}_{W_x}(-1).$$

Hence we have $-\lambda(W_x|_{\widetilde{C}}) = \mathcal{O}_{\widetilde{C}}(\lambda)$. Since E is nef on X by hypothesis, it follows that $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x|_{\widetilde{C}})$ is nef for every $\lambda > 0$.

Now suppose that $\widetilde{C}$ is not contained in W_x, and let $C = \varpi(\widetilde{C})$. Then $C \subset X$ is a T–invariant curve. If $x \notin C$, then $W_X|_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}}$, and $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1)$ is nef because $\varpi^\star E|_{\widetilde{C}}$ is so.

So assume that $x \in C$. Then $W_x \cdot \widetilde{C} = 1$, since C is a smooth curve. Let $a_1(C), \cdots, a_n(C)$ be non-negative integers such that $E|_C = \mathcal{O}_C(a_1(C)) \oplus \ldots \oplus \mathcal{O}_C(a_n(C))$. Then $\mathcal{O}_{\mathbb{P}(\varpi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x|_{\widetilde{C}})$ is nef if and only if $\mathcal{O}_{\widetilde{C}}(a_1(C) - \lambda) \oplus \cdots \oplus \mathcal{O}_{\widetilde{C}}(a_n(C) - \lambda)$ is nef. Now. $\mathcal{O}_{\widetilde{C}}(a_1(C) - \lambda) \oplus \cdots \oplus \mathcal{O}_{\widetilde{C}}(a_n(C) - \lambda)$ is nef if and only if $\lambda \leq \min\{a_1(C), \cdots, a_n(C)\}$. Running over all T–invariant curves in $\widetilde{X}$ the theorem is proved. $\square$

Remark 3.4. A similar computation of Seshadri constants was carried out for equivariant vector bundles on toric varieties in [HMP, Proposition 3.2] which motivated our result.

The following is well-known, but we give this example to show how our results give a simpler argument.

Example 3.5. Let $0 < d < n$ be integers and let $X = \mathrm{Gr}(d, n)$ be the Grassmannian of d–dimensional subspaces of $\mathbb{C}^n$. Then one has the universal exact sequence

$$0 \longrightarrow S \longrightarrow X \times \mathbb{C}^n \longrightarrow Q \longrightarrow 0$$

of vector bundles on X, where the fiber of S over a point $x \in X$ is the d–dimensional subspace S_x of $\mathbb{C}^n$ corresponding to $x \in X$ while the fiber of Q over x is the quotient vector space $\mathbb{C}^n/S_x$. Further, it is easy to prove that all three vector bundles in the above exact sequence are $\mathrm{GL}_n(\mathbb{C})$–equivariant, and hence they are T–equivariant, where $T \subset \mathrm{GL}_n(\mathbb{C})$ is the subgroup of diagonal matrices.

Now let $C \subset X$ be a T–invariant curve. Then $Q|_C = \mathcal{O}_C(1) \oplus \mathcal{O}_C^{\oplus(n-d-1)}$. Note that $C \cong \mathbb{CP}^1$. Hence $Q|_C$ is always nef and it is ample if and only if $n - d = 1$. So, by Theorem 3.1, the vector bundle Q itself is always nef and it is ample if and only if $n - d = 1$. Moreover, if Q is ample, then $\varepsilon(Q, x) = 1$ for every T–fixed point $x \in X$ by Theorem 3.3.

The tangent bundle $\mathcal{T}_X$ is isomorphic to $\mathcal{H}om(S, Q) = S^\vee \otimes Q$, where $S^\vee$ denotes the dual of S. Arguing as above, we conclude that $\mathcal{T}_X$ is ample if and only if either S or Q is a line bundle which is the case precisely when $d = 1$ or $n - d = 1$. Of course, this happens if and only if X is the projective space $\mathbb{CP}^n$.

Finally note that the determinant bundle $\det(Q)$ is T–equivariant, because Q is so. If C is any T–invariant closed curve in X, then $\det(Q)|_C = \mathcal{O}_C(1)$. Hence the vector bundles $Q \otimes \det(Q)$ and $S^\vee \otimes \det(Q)$ are both ample.

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