POSITIVITY OF LINE BUNDLES ON GENERAL BLOW UPS OF $\mathbb{P}^2$

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ABSTRACT. Let X be the blow up of $\mathbb{P}^2$ at r general points $p_1, \ldots, p_r \in \mathbb{P}^2$. We study line bundles on X given by plane curves of degree d passing through p_i with multiplicity at least m_i. Motivated by results in [ST3], we establish conditions for ampleness, very ampleness and global generation of such line bundles.

1. INTRODUCTION

Let $p_1, p_2, \ldots, p_r$ be general points of $\mathbb{P}^2$. We consider the blow up $\pi : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $p_1, \ldots, p_r$. Let $H = \pi^\star \mathcal{O}_{\mathbb{P}^2}(1)$. Let $E_1, E_2, \ldots, E_r$ denote the exceptional divisors of the blow up.

Line bundles on X of the form $L = dH - \sum_{i=1}^r m_i E_i$ where d is a positive integer and $m_1, \ldots, m_r$ are non-negative integers are extensively studied. These line bundles are associated to the linear system of curves in $\mathbb{P}^2$ of degree d passing through the point p_i with multiplicity at least m_i, for $1 \leqslant i \leqslant r$. Such linear systems are of great interest and they are central to many important questions in algebraic geometry. One important problem concerns the positivity properties of these line bundles. Specifically, one asks if such line bundles are ample, globally generated or very ample. There are also various generalizations of these notions which are studied in this context.

Several authors have addressed these questions in different situations.

Let $L = dH - \sum_{i=1}^r m_i E_i$. First suppose that the points $p_1, \ldots, p_r$ are general. The "uniform" case, where m_i are all equal, has been well-studied. Küchle [K] and Xu [X2] independently consider the case $m_i = 1$ for all i. In this case, they in fact prove that L is ample if and only if $L^2 > 0$, when $d \geqslant 3$. See [K, Corollary] and [X2, Theorem]. [Bi] considers the case $m_i = 2$ for all i and gives a similar criterion (see [Bi, Remark, Page 120]). In this situation, conditions for nefness of L are given in [H8].

[BC] gives some technical conditions for ampleness and very ampleness in general for the blow up of any projective variety at distinct points.

[Be, DG] treat the case $m_i = 1$ for all i for *arbitrary* points $p_1, \ldots, p_r$. [Be] gives conditions for global generation and very ampleness. In [Be, Theorem 2], it is proved that L is globally generated if $d \geqslant 3$, $r \leqslant \frac{\binom{d+5}{2}-4}{3}$ and at most $k(d+3-k)-2$ of the points p_i lie on

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a curve of degree k, where $1 \leqslant k \leqslant \frac{d+3}{2}$. A similar result is given for very ampleness in [Be, Theorem 3]. [DG] also gives conditions for global generation and very ampleness.

When the points $p_1, \ldots, p_r$ are general, [AH] considers the case $m_i = 1$ for all i and gives a condition for very ampleness. It is proved that L is very ample when $r \leqslant \frac{d^2+3d}{2} - 5$ ([AH, Theorem 2.3]). This result is generalized to arbitrary projective space $\mathbb{P}^n$ in [C, Theorem 1]. A converse to [AH, Theorem 2.3] is proved and a complete characterization for very ampleness is given in [GGP, Proposition 3.2] when $m_i = 1$ for all i.

When $p_1, \ldots, p_r$ are arbitrary points on a smooth cubic in $\mathbb{P}^2$, [H2] gives conditions for ampleness and very ampleness. The assumption on the points means that the anticanonical class is effective, making the variety *anticanonical*. In [H2, Theorem 1.1], a criterion for ampleness is given in terms of intersection with a small set of exceptional and nodal classes. In [H2, Theorem 2.1], L is proved to be very ample if and only if it is ample and its restriction to the anticanonical class is very ample. These results were generalized in [DT] by giving conditions for k-*very ampleness* of L when p_i are general points on a smooth cubic. [H5] studies an anticanonical surface X in general and describes base locus of line bundles on X and in the process characterizes when these are globally generated.

These questions can also be asked when X is a blow up of points on surfaces other than $\mathbb{P}^2$. [ST2] studies this question for $\pi : X \to S$, where S is an abelian surface and X is the blow up of S at r general points. They obtain conditions for very ampleness, and more generally for k-very ampleness, of $\pi^\star(L) - \sum_{i=1}^{r} E_i$ where L is a polarization on S and E_i are the exceptional divisors. [ST1] considers this problem when S is a ruled surface.

In this paper we consider blow ups of $\mathbb{P}^2$ at general points and obtain sufficient conditions for ampleness, global generation and very ampleness. Main motivation for us comes from the work of Szemberg and Tutaj-Gasińska in [ST3]. They consider the case $2 \leqslant m = m_i$ for all i and show that $L = dH - m\sum_{i=1}^{r} E_i$ is ample if $d \geqslant 3m + 1$ and $d^2 \geqslant (r+1)m^2$ ([ST3, Theorem 3]). They also establish conditions for k-very ampleness of L. In addition to improving their bounds, we consider the non-uniform case and give conditions for ampleness and global generation.

Our first main result is Theorem 2.1 and it gives conditions for ampleness for an arbitrary line bundle $L = dH - \sum_{i=1}^{r} m_i E_i$. We then discuss the *SHGH Conjecture* (see Section 2). Our second main result is Theorem 2.18 which gives better conditions for ampleness for a uniform line bundle by using known cases of the SHGH Conjecture. Theorem 3.1 gives conditions for global generation of an arbitrary line bundle and Theorem 3.5 deals with global generation for uniform line bundles. Theorem 3.7 provides conditions for very ampleness in the uniform case.

Note that a necessary condition for ampleness of $L = dH - \sum_{i=1}^{r} m_i E_i$ is $L^2 > 0$. This condition is equivalent to $d^2 > \sum_{i=1}^{r} m_i^2$. In general this is not sufficient, but for $r \geqslant 9$, this condition is conjectured to be sufficient in the uniform case. See the *Nagata Conjecture* in Section 2. In this paper we prove ampleness under the assumption: $d^2 \geqslant c_r \sum_{i=1}^{r} m_i^2$, where $c_r > 1$ depends on r. We give several results with different values of c_r. In Theorem 2.1, we give a result for arbitrary line bundles with $c_r = \frac{r+3}{r+2}$. In uniform case we get a better bound $c_r = \frac{3r+40}{3r+39}$. See Theorem 2.18. Though our bounds are not optimal, we do obtain new cases. We compare our results with some existing results. We also give several examples to illustrate our results.

The key ingredient in our arguments is a result of Xu [X1, Lemma 1] and Ein-Lazarsfeld [EL]. This result, using deformation techniques, gives a lower bound for the degree of a plane curve (over $\mathbb{C}$) passing through a finite set of general points with prescribed multiplicities. This bound is improved in [KSS, Theorem A]. Ampleness (by Nakai-Moishezon), global generation and very ampleness (by Reider) can be tested by looking at intersection numbers with effective curves. We bound these intersection numbers by giving suitable conditions and using the lower bound of Xu and Ein-Lazarsfeld.

In Section 2 we give conditions for ampleness. Conditions for global generation and very ampleness are studied in Section 3.

We work throughout over the complex number field $\mathbb{C}$. When we say that $p_1, \ldots, p_r$ are *general points* of $\mathbb{P}^2$, we mean that they belong to an open dense subset of $(\mathbb{P}^2)^r$. More precisely, if a statement holds for general points $p_1, \ldots, p_r \in \mathbb{P}^2$, it holds for all points in an open dense subset of $(\mathbb{P}^2)^r$.

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2. AMPLENESS

We first give conditions for ampleness of a line bundle on X.

Theorem 2.1. *Let $p_1, p_2, \ldots, p_r$ be $r \geqslant 1$ general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ not passing through any of the points p_i and by $E_1, \ldots, E_r$ the exceptional divisors. Let L be the line bundle $dH - \sum_{i=1}^{r} m_i E_i$ for some $d > 0$ and $m_1 \geqslant \ldots \geqslant m_r > 0$. Then L is ample if the following conditions hold:*

(1) $d > m_1 + m_2$,
(2) $2d > m_1 + m_2 + \ldots + m_5$,
(3) $3d > 2m_1 + m_2 + \ldots + m_7$, *and*
(4) $d^2 \geqslant \frac{s+3}{s+2} \sum_{i=1}^{s} m_i^2$, *for* $2 \leqslant s \leqslant r$.

Proof. We use the Nakai-Moishezon criterion for ampleness. We have $L^2 = d^2 - \sum_{i=1}^{r} m_i^2 > 0$, by hypothesis (4). We show below that $L \cdot C > 0$ for any irreducible, reduced curve C on X.

Let $C \subset X$ be such a curve. First we consider the case $C = \sum_{i=1}^{r} n_i E_i$ for some $n_i \in \mathbb{Z}$, not all zero. If each n_i is non-positive, then C can not be effective, being negative of an effective curve. On the other hand, if $n_i > 0$ for some i, then $C \cdot E_i = -n_i < 0$. Since C and E_i are both reduced and irreducible, it follows that $C = E_i$. Then $L \cdot C = m_i > 0$.

We now assume that C is in the linear system of a line bundle $eH - \sum_{i=1}^{r} n_i E_i$ for some positive integer e and non-negative integers $n_1, \ldots, n_r$ with $n_1 + \ldots + n_r > 0$. We may further arrange n_i in decreasing order: $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. Choose $s \in \{1, \ldots, r\}$ such that $n_s \neq 0$ and $n_{s+1} = \ldots = n_r = 0$.

Then C is the strict transform of a reduced and irreducible curve of degree e in $\mathbb{P}^2$ which passes through p_i with multiplicity n_i, for every i.

Then by [X1, Lemma 1] or [EL], we have

$$e^2 \geqslant \sum_{i=1}^{r} n_i^2 - n_s. \tag{2.1}$$

Along with hypothesis (4) of the theorem, this gives

$$d^2e^2 \geqslant \frac{r+3}{r+2}\left(\sum_{i=1}^{r} m_i^2\right)\left(\sum_{i=1}^{r} n_i^2 - n_s\right). \tag{2.2}$$

We now consider different cases.

Case A: Let $n_1 \geqslant 2$.

We first suppose that $s = r$.

If $r = 2$ and $n_1 = n_2 = 2$, then $e \geqslant 3$. So hypothesis (1) gives $L \cdot C > 0$. We assume henceforth that if $r = 2$, then $(n_1, n_2) \neq (2, 2)$.

Then, by (2.2) and Lemma 2.3 below, $d^2e^2 \geqslant \left(\sum_{i=1}^{r} m_i n_i\right)^2$. Hence $de \geqslant \sum_{i=1}^{r} m_i n_i$ and $L \cdot C \geqslant 0$. We also know that the inequality in Lemma 2.3 is strict except in two cases which we treat below. In addition, if $de = \sum_{i=1}^{r} m_i n_i$, we must have an equality in (2.1). So $e^2 = \sum_{i=1}^{r} n_i^2 - n_r$.

If $(n_1, \ldots, n_r) = (2, 1, \ldots, 1)$ then $e^2 = r+2$. If $r = 2$ then $C = 2H - 2E_1 - E_2$ and $L \cdot C > 0$ by hypothesis (1). If $r = 7$, then $C = 3H - 2E_1 - E_2 - \ldots - E_7$ and $L \cdot C > 0$ by hypothesis (3). For $r = 14$, there is no curve of degree 4 passing through p_1 with multiplicity 2 and through $p_2, \ldots, p_{14}$ with multiplicity 1. For, a point of multiplicity 2 imposes 3 conditions and 13 simple general points impose another 13 conditions. So there are a total of 16 conditions, while the space of plane curves of degree 4 has only dimension $\binom{6}{2} - 1 = 14$. Same argument holds when $r > 14$.

We get an equality in Lemma 2.3 also when $r = 3$ and $n_1 = n_2 = n_3 = 2$. But in this case $L \cdot C > 0$ by hypothesis (1). Indeed, by (1), $2d > 2m_1 + 2m_2 \geqslant 4m_3 \Longrightarrow d > 2m_3 \Longrightarrow 3d > 2m_1 + 2m_2 + 2m_3$.

Thus in all cases $L \cdot C > 0$ as desired.

Next suppose that $s < r$. If $s = 1$ then $e \geqslant n_1$. Since $d > m_1$, by hypothesis (1), we get $de > m_1 n_1$.

Let $s \geqslant 2$. By hypothesis (4), $d^2 \geqslant \frac{s+3}{s+2} \sum_{i=1}^{s} m_i^2$. So as above (applying Lemma 2.3 with $r = s$), it follows that $de > \sum_{i=1}^{s} m_i n_i$. Thus $L \cdot C > 0$.

Case B: Let $C = eH - E_1 - E_2 - \ldots - E_s$.

We have $e^2 \geqslant s-1$. For $s \leqslant 5$, hypotheses (1) and (2) show that $L \cdot C > 0$. Let $s \geqslant 6$, so that $e \geqslant 3$. We claim that $e^2 \geqslant s$. Indeed, the space of plane curves of degree e has dimension $\binom{e+2}{2} - 1$ and each point p_i imposes one condition. So $\binom{e+2}{2} - 1 \geqslant s$. It is easy to see that $e^2 \geqslant \binom{e+2}{2} - 1$ when $e \geqslant 3$, so that $e^2 \geqslant s$ as claimed.

By hypothesis (4), $d^2 > \sum_{i=1}^{s} m_i^2$. The following is a standard inequality:

$$s(m_1^2 + \ldots + m_s^2) \geqslant (m_1 + \ldots + m_s)^2.$$

So $d^2 > \frac{(\sum_{i=1}^{s} m_i)^2}{s}$. Hence $(de)^2 \geqslant d^2 s > (\sum_{i=1}^{s} m_i)^2$.

We conclude that L is ample. □

Remark 2.2. The hypotheses (1)-(3) of Theorem 2.1 are necessary in order for L to be ample. This is due to the existence, respectively, of a line through two general points, a conic through five general points and a cubic through seven general points with one of the points being a double point and all others simple points.

Lemma 2.3. *Let* $r \geqslant 2$, $m_1, m_2 \ldots, m_r \geqslant 0$ *and* $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r > 0$. *Suppose that*

- $n_1 \geqslant 2$, *and*
- *if* $r = 2$ *then* $(n_1, n_2) \neq (2, 2)$.

Then the following inequality holds:

$$\frac{r+3}{r+2}\left(\sum_{i=1}^{r} m_i^2\right)\left(\sum_{i=1}^{r} n_i^2 - n_r\right) \geqslant \left(\sum_{i=1}^{r} m_i n_i\right)^2.$$

Moreover, the inequality is strict when $(n_1, n_2, \ldots, n_r) \neq (2, 1, \ldots, 1)$ *and if* $r = 3$, $(n_1, n_2, n_3) \neq (2, 2, 2)$.

Proof. For simplicity, let $a = \sum_{i=1}^{r} m_i^2, b = \sum_{i=1}^{r} n_i^2$ and $c = \sum_{i=1}^{r} m_i n_i$. So the desired inequality is $\frac{r+3}{r+2} a(b - n_r) \geqslant c^2$.

Rearranging terms and clearing the denominator, the desired inequality is equivalent to $(r+3)ab - (r+2)c^2 \geqslant (r+3)n_r a$, which in turn is equivalent to

$$ab + (r+2)(ab - c^2) \geqslant (r+3)n_r a. \tag{2.3}$$

Now we apply the well-known equality:

$$ab - c^2 = \left(\sum_{i=1}^{r} m_i^2\right)\left(\sum_{i=1}^{r} n_i^2\right) - \left(\sum_{i=1}^{r} m_i n_i\right)^2 = \sum_{i<j} (m_i n_j - m_j n_i)^2. \tag{2.4}$$

The left hand side of (2.3) is then $ab + (r+2)\sum_{i<j}(m_i n_j - m_j n_i)^2$ and it is at least ab.

We now show the following inequality from which (2.3) and the lemma follow.

$$b \geqslant (r+3)n_r. \tag{2.5}$$

If $n_r = 1$, then the right hand side of (2.5) is $r+3$. When $n_r = 1$ the left hand side of (2.5) is smallest when $(n_1, \ldots, n_r) = (2, 1, \ldots, 1)$ and the minimum in this case is $r+3$.

Suppose now that $n_r > 1$. Then minimum of the left hand side of (2.5) is obtained when $n_1 = n_2 = \ldots = n_r$ and it is rn_r^2. To finish, note that $rn_r^2 \geqslant (r+3)n_r$ is equivalent to $r(n_r - 1) \geqslant 3$. This fails only when $r = 2$ and $n_r = 2$ which we omitted in the lemma. Further it is an equality only when $r = 3$ and $n_r = 2$. In all other cases we have a strict inequality and the lemma follows. $\square$

Example 2.4. Let $L_d = dH - 3E_1 - \sum_{i=2}^{8} 2E_i - \sum_{i=9}^{12} E_i$. Then the smallest d which satisfies the hypotheses (1)-(4) of Theorem 2.1 is $d = 7$: d must satisfy $d > 5$, $2d > 11$, and $3d > 18$. It is clear that $d = 7$ satisfies hypothesis (4). For instance $7^2 > \frac{15}{14}(41) = 43.92$. So $L_7 = 7H - 3E_1 - \sum_{i=2}^{8} 2E_i - \sum_{i=9}^{12} E_i$ is ample. If $d < 7$, $L_d^2 < 0$ and thus L_d is not ample. So we get the optimal result in this example.

Example 2.5. Let $L_d = dH - 3E_1 - \sum_{i=2}^{10} 2E_i - \sum_{i=11}^{12} E_i$. Then the smallest d which satisfies the hypotheses (1)-(4) of Theorem 2.1 is $d = 8$. By hypothesis (4), we require $d^2 \geqslant \frac{15}{14}(47) = 50.35$. It is easy to see that $d = 8$ satisfies all the other hypotheses. So $L_8 = 8H - 3E_1 - \sum_{i=2}^{10} 2E_i - \sum_{i=11}^{12} E_i$ is ample. But $L_7^2 > 0$ and it is ample if the SHGH Conjecture, which is recalled below, is true. See also Example 2.21. Our result in this case is thus not expected to be optimal.

Example 2.6. Consider $L_d = dH - 10\sum_{i=1}^{5} E_i$. Then L_{25} is not ample, because there is a conic through 5 general points and $L_{25} \cdot (2H - \sum_{i=1}^{5} E_i) = 0$. Note that $L_{25}^2 > 0$, showing that $L^2 > 0$ is not a sufficient condition for ampleness when $r = 5$. On the other hand, L_{26} is ample by Theorem 2.1.

Question 2.7. Does the conclusion of Theorem 2.1 hold with the hypotheses (1)-(3) and only that $d^2 \geqslant \frac{r+3}{r+2}\sum_{i=1}^{r} m_i^2$ (that is, we only require the hypothesis (4) for $s = r$)? Our calculations suggest that the answer is yes, but we do not have a proof.

However, we have the following theorem which says that hypothesis (4) need only be checked for $s \geqslant 9$, if we require that L meets *exceptional curves* positively.

Let C be a curve on X. We say that C is an *exceptional curve* or a (-1)-*curve* if it is a smooth rational curve such that $C^2 = -1$. If C is a (-1)-curve, by adjunction, we also have $C \cdot K_X = -1$, where $K_X = -3H + \sum_{i=1}^{r} E_i$ is the canonical line bundle on X.

Proposition 2.8. *Let $p_1, p_2, \ldots, p_r$ be $r \geqslant 9$ general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ and by $E_1, \ldots, E_r$ the exceptional divisors. Let L be the line bundle on X given by $dH - \sum_{i=1}^{r} m_i E_i$ for $d > 0$ and $m_1 \geqslant \ldots \geqslant m_r > 0$. Then L is ample if the following conditions hold:*

(1) $d > m_1 + m_2$,
(2) $2d > m_1 + m_2 + \ldots + m_5$,
(3) $3d > 2m_1 + m_2 + \ldots + m_7$,
(4) $d^2 \geqslant \frac{s+3}{s+2}\sum_{i=1}^{s} m_i^2$, *for* $9 \leqslant s \leqslant r$, *and*
(5) $L \cdot E > 0$, *where E is one of the following exceptional curves*
$4H - 2E_1 - 2E_2 - 2E_3 - E_4 - \ldots - E_8$, *or*
$5H - 2E_1 - \ldots - 2E_6 - E_7 - E_8$, *or*
$6H - 3E_1 - 2E_2 - \ldots - 2E_8$.

Proof. The proof is exactly as in Theorem 2.1. The difference here is that hypothesis (4) is required for only $s \geqslant 9$. For $s < 9$, we use the following fact.

Let C be a curve in X which belongs to the linear system of a divisor of the form $eH - n_1E_1 - \ldots - n_sE_s$ with $s < 9$, $e > 0$ and each $n_i \geqslant 0$. The blow up $X \to \mathbb{P}^2$ factors as $X \to X_1 \to \mathbb{P}^2$ where X_1 is the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_s$. Then C is isomorphic to its image in X_1. Denote this image also by C. Then the class of C in the Picard lattice of X_1 is a linear combination of a numerically effective class and a finite set of exceptional classes. See [H4, Theorem 1(a)] for a proof of this fact. Hence the class of C in the Picard lattice of X is also such a linear combination.

Moreover there are only finitely many exceptional classes of curves coming from plane curves passing through less than 9 points. The proof of [N2, Theorem 4a, Page 284] lists these exceptional classes (see also Discussion 2.13). Of the seven classes listed there three correspond to the classes in hypothesis (5). Hypotheses (1)-(3) and the hypothesis $m_i > 0$ account for the other exceptional curves in the list.

Now if an irreducible and reduced curve C is numerically effective then by Lemma 2.10 below it follows that $L \cdot C > 0$. If C is an exceptional curve then $L \cdot C > 0$ by hypothesis. □

Remark 2.9. In [X3, Theorem 2], Xu proves a result comparable to our Theorem 2.1. In addition to some necessary conditions to ensure L meets exceptional curves positively, the main hypothesis for [X3, Theorem 2] is $d^2 > \frac{10}{9}\sum_{i=1}^r m_i^2$. This result follows from Proposition 2.8 because $\frac{s+3}{s+2} \leqslant \frac{10}{9}$ when $s \geqslant 7$.

We make the following useful observation.

Lemma 2.10. *Let $L = dH - \sum_{i=1}^r m_iE_i$ be a line bundle on X such that $L^2 > 0$. Let C be an irreducible and reduced curve on X with $C^2 \geqslant 0$. Then $L \cdot C > 0$.*

Proof. Write $C = eH - \sum_{i=1}^r n_iE_i$. We may assume that $e > 0$ and $n_i \geqslant 0$ for every i. Since $C^2 \geqslant 0$, we have $e^2 \geqslant \sum_{i=1}^r n_i^2$. Since $L^2 > 0$, we have $d^2 > \sum_{i=1}^r m_i^2$. Thus $d^2e^2 > \left(\sum_{i=1}^r m_i^2\right)\left(\sum_{i=1}^r n_i^2\right) \geqslant \left(\sum_{i=1}^r m_in_i\right)^2$. The last inequality follows from (2.4). Thus $L \cdot C > 0$. □

Thus a line bundle L with $L^2 > 0$ fails to be ample only if there is a curve C such that $C^2 < 0$ and $L \cdot C < 0$.

When $m_i = m$ for all i (uniform case), Question 2.7 has a positive answer. Hypothesis (4) of Theorem 2.1 is then simply $d^2 \geqslant \frac{r(r+3)}{r+2}m^2$. Indeed, since $\frac{r+3}{r+2} \geqslant \frac{s+3}{s+2}$ for any $s \leqslant r$, it automatically follows that $d^2 \geqslant \frac{s(s+3)}{s+2}m$ for any $2 \leqslant s \leqslant r$.

In fact, we have the following corollary in the uniform case.

For $r \geqslant 2$, we define λ_r as follows: $\lambda_2 = \lambda_3 = 2$, $\lambda_5 = 2.5$ and for all other r, $\lambda_r = \sqrt{\frac{r(r+3)}{r+2}}$.

Proposition 2.11. *Let $r \geqslant 2$. Let $L = dH - m\sum_{i=1}^r E_i$ be a line bundle on X with $m > 0$. Suppose that $d > \lambda_r m$. Then L is ample.*

Proof. Let $C = eH - \sum_{i=1}^r n_iE_i$ be an irreducible, reduced curve on X. We show that $L \cdot C > 0$.

We may assume that $e > 0$. Write $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. We assume without loss of generality that $n_r > 0$.

When $n_1 = 1$ it is clear that $L \cdot C > 0$, as in **Case B** in the proof of Theorem 2.1. So we assume $n_1 \geqslant 2$.

For $r = 2, 3, 5$ the bound in hypothesis (4) of Theorem 2.1 is lower than the necessary conditions contained in the hypotheses (1)-(3) of that theorem. These numbers are respectively, 2, 2 and 2.5 for $r = 2, 3, 5$, which we defined to be λ_2, λ_3 and λ_5. So we have ampleness in these cases.

Now we apply Lemma 2.3 with $m_i = m$ for all i. Thus

$$\frac{(r+3)r}{r+2}m^2\left(\sum_{i=1}^r n_i^2 - n_r\right) \geqslant m^2\left(\sum_{i=1}^r n_i\right)^2.$$

Rearranging terms we get $\sum_{i=1}^r n_i^2 - n_r \geqslant \frac{r+2}{r(r+3)}(\sum_{i=1}^r n_i)^2$. So $e \geqslant \sqrt{\frac{r+2}{r(r+3)}}(\sum_{i=1}^r n_i) = \frac{1}{\lambda_r}(\sum_{i=1}^r n_i)$. Thus $de > m\sum_{i=1}^r n_i$. □

Example 2.12. Consider $r = 8$ and the line bundle $L_d = dH - 60\sum_{i=1}^8 E_i$. Note that $L_{169}^2 = 28561 - 28800 = -239 < 0$ and $L_{170}^2 = 28900 - 28800 > 0$.

By [N2] (see the Introduction to [H6]) there is a degree 48 curve in $\mathbb{P}^2$ passing through 8 general points with multiplicity 17 at each of the 8 points. In the notation of [H6], $\delta(m, n)$ is the least integer such that there is a curve of that degree in $\mathbb{P}^2$ passing through n general points with multiplicity at least m. For $n \leqslant 9$, $\delta(m, n)$ is the ceiling of $c_n m$ where c_n is an explicitly determined rational number. It turns out that $c_8 = \frac{48}{17}$ by [N2].

The strict transform of such a curve on the blow up of the 8 points is a curve C in the class of $48H - 17\sum_{i=1}^8 E_i$. Then $(170H - 60\sum_{i=1}^8 E_i) \cdot (48H - 17\sum_{i=1}^8 E_i) = 0$. So L_{170} is not ample. An exceptional curve in the base locus of the linear system of C is $E = 6H - 3E_1 - 2E_2 - \ldots - 2E_8$. It is easily checked that $L_{170} \cdot E = 0$. We can verify that E is indeed an exceptional curve by using Nagata's result that exceptional curves form a single orbit under a group action and the fact that E is a translate of E_8 under this group action. See Discussion 2.13 and the proof of Lemma 2.14.

We see that L_{178} is ample by Proposition 2.11. For, $\lambda_8 = 2.9664$ and $\lambda_8 m = 177.9887$. We consider this example again in Example 2.20 and show in fact that L_{171} is ample.

Our goal now is to obtain better bounds in the uniform case. We first recall the so-called *SHGH Conjecture*.

Let $p_1, \ldots, p_r$ be general points in $\mathbb{P}^2$ and let $d, m_1, \ldots, m_r$ be non-negative integers. Let L be the linear system of degree d curves in $\mathbb{P}^2$ passing through p_i with multiplicity at least m_i. Then L corresponds to a line bundle $dH - \sum_{i=1}^r m_iE_i$ on X, where X is the blow up of $\mathbb{P}^2$ at the points p_i. We denote this line bundle also by L.

Note that the projective space of plane curves of degree d has dimension $\binom{d+2}{2} - 1$ and a point of multiplicity m_i imposes $\binom{m_i+1}{2}$ conditions. So we say that the *expected dimension* of the linear system L is $\binom{d+2}{2} - 1 - \sum_{i=1}^{r} \binom{m_i+1}{2}$ if this number is non-negative and -1 otherwise. The actual dimension of L is greater than or equal to the expected dimension. The linear system L is called *special* if its actual dimension is more than the expected dimension.

By Riemann-Roch theorem L is special if and only if $h^0(X, L)h^1(X, L) \neq 0$.

Easy examples of special linear systems are given by (-1)-curves. If C is a (-1)-curve, then the linear system of $2C$ is special. Indeed, by Riemann-Roch, we have $h^0(2C) - h^1(2C) = \frac{2C\cdot(2C-K)}{2} + 1$. Note that $h^2(2C) = 0$ since $2C$ is effective. Also, $2C \cdot (2C - K) = 4C^2 - 2C \cdot K = -4 + 2 = -2$. So $h^0(2C) - h^1(2C) = 0$, which implies that $h^0(2C)$ and $h^1(2C)$ are both nonzero. In fact, they are both equal to 1. More generally, any linear system with a (-1)-curve in its base locus (with multiplicity at least 2) is special.

The SHGH conjecture predicts that every special linear system arises from (-1)-curves as above. This conjecture was formulated by Segre [S], Harbourne [H3], Gimigliano [G] and Hirschowitz [Hi]. There are different formulations of this conjecture. See [H7] for a nice survey. [CM3] shows equivalence of various versions.

The following is one version of the SHGH Conjecture.

SHGH Conjecture. *Let X be the blow up of $\mathbb{P}^2$ at r general points. Then the following statements hold.*

(1) *Any reduced, irreducible curve on X with negative self-intersection is a (-1)-curve;*
(2) *Any nef and effective linear system is non-special.*

See [CM1, CM2, M, dF, Y, DJ, CHMR] for some progress on this conjecture. In particular, [DJ, Theorem 34] verifies the SHGH Conjecture when $m_i \leqslant 11$ for all i. We also note that [dF, Theorem 2.5] verifies Statement (1) of the SHGH Conjecture when one of the multiplicities is 2, that is, $m_i = 2$ for some i.

Discussion 2.13. Let $r \geqslant 3$. For the divisor class group $\mathrm{Cl}(X)$ of X, consider the linear map $\gamma_0 : \mathrm{Cl}(X) \to \mathrm{Cl}(X)$, given by:

$\gamma_0(H) = 2H - E_1 - E_2 - E_3$,

$\gamma_0(E_i) = H - E_1 - E_2 - E_3 + E_i$ for $i = 1, 2, 3$, and

$\gamma_0(E_i) = E_i$ for $i \geqslant 4$.

For $i = 1, 2, \ldots, r-1$, let $\gamma_i : \mathrm{Cl}(X) \to \mathrm{Cl}(X)$ be the map which interchanges E_i and E_{i+1} and fixes H and E_j when $j \notin \{i, i+1\}$.

Let W_r denote the group of linear automorphisms of $\mathrm{Cl}(X)$ generated by $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$. There is a root system on $\mathrm{Cl}(X)$ with simple roots given by

$$s_0 = H - E_1 - E_2 - E_3, s_i = E_i - E_{i+1}, \text{ for } 1 \leqslant i \leqslant r-1.$$

The maps γ_i may be regarded as reflections orthogonal to the simple roots. More concretely, for any $L \in Cl(X)$ and i, $\gamma_i(L) = L + (L \cdot s_i)s_i$. In this point of view, W_r is the Weyl group of the root system on $Cl(X)$ with simple roots $s_0, \ldots, s_{r-1}$. See [L2] and [H1] for more details.

Lemma 2.14. *Let $r \geqslant 3$. Let C be an exceptional curve on X given by $eH - \sum_{i=1}^{r} n_i E_i$ with $e > 0$ and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. Then $e < n_1 + n_2 + n_3$.*

Proof. Nagata [N2] proved that exceptional curves on X form a single orbit under the action of W_r on $Cl(X)$, when $r \geqslant 3$. We consider the fundamental domain for the action of W_r on $Cl(X)$. This consists of line bundles with non-negative intersection with all simple roots. Since the exceptional curves form a single W_r-orbit, exactly one exceptional curve is in the fundamental domain. It is clear that E_r is in the fundamental domain, since it has non-negative intersection with all simple roots: $E_r \cdot s_i \geqslant 0$ for $i = 1, \ldots, r$.

Since $C \neq E_r$, it is not in the fundamental domain. So $C \cdot s_i < 0$ for some simple root s_i. Since $n_1 \geqslant \ldots \geqslant n_r$, C meets the roots $s_1, \ldots, s_r$ non-negatively. Hence $C \cdot s_0 = e - n_1 - n_2 - n_3 < 0$. □

Proposition 2.15. *Let $L = dH - m\sum_{i=1}^{r} E_i$ be a line bundle on X with $L^2 > 0$. Suppose that there exists a positive integer N such that the SHGH Conjecture holds for all curves of the form $eH - \sum_{i=1}^{r} n_i E_i$ with $0 \leqslant n_i \leqslant N$ for all i. Suppose further that L meets positively all the finitely many exceptional curves with multiplicity at most N at p_i for all i.*

Now let $C = eH - \sum_{i=1}^{r} n_i E_i$ with $0 \leqslant n_i \leqslant N$ be an irreducible and reduced curve in X. Then $L \cdot C > 0$.

Proof. If $C^2 < 0$, then the SHGH Conjecture implies that C is a (-1)-curve. So $L \cdot C > 0$ by hypothesis. If $C^2 \geqslant 0$, we are done by Lemma 2.10. □

Remark 2.16. The SHGH Conjecture is known to be true for curves with multiplicities up to 11, by [DJ, Theorem 34]. So the result of Proposition 2.15 holds with $N = 11$.

Next we will prove a lemma similar to Lemma 2.3.

Lemma 2.17. *Let $r \geqslant 9$ and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 3$. Suppose that $n_1 > 11$. Then the following inequality holds:*

$$\frac{(3r+40)r}{3r+39}\left(\sum_{i=1}^{r} n_i^2 - n_r\right) > \left(\sum_{i=1}^{r} n_i\right)^2. \tag{2.6}$$

Proof. This is similar to the proof of Lemma 2.3. Let $a = \sum_{i=1}^{r} n_i^2$ and $b = \sum_{i=1}^{r} n_i$. Clearing the denominator and rearranging terms, the desired inequality is $ra + (3r + 39)(ra - b^2) > r(3r + 40)n_r$. Since $ra - b^2 \geqslant 0$ the lemma will follow if $ra > r(3r + 40)n_r$, or equivalently, if $a > (3r + 40)n_r$.

For $n_r = 3$, the smallest value of a is obtained when $(n_1, \ldots, n_r) = (12, 3, \ldots, 3)$ and then $a = 144 + 9r - 9 = 9r + 135$ which is clearly greater than $3(3r + 40)$. When $n_r = 4$, the smallest value of a is $16r + 128$ and it is obtained when $(n_1, \ldots, n_r) = (12, 4, \ldots, 4)$. It is easy to see that we have $16r+128 > 4(3r+40)$ for $r \geqslant 9$. When $n_r = 5$, the smallest value of a is $25r+119$ attained when $(n_1,\ldots,n_r) = (12,5,\ldots,5)$ and again it is easy to verify that $25r+119 > 5(3r+40)$ for $r \geqslant 9$. In exactly the same way, the required inequality follows when $n_r \geqslant 6$. □

Now we are ready to prove our main result about ampleness in the uniform case.

Theorem 2.18. *Let $p_1, p_2, \ldots, p_r$ be $r \geqslant 1$ general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ not passing through any of the points p_i and by $E_1, \ldots, E_r$ the exceptional divisors. Let $L = dH - m\sum_{i=1}^r E_i$ be a line bundle on X with $m > 0$. Then L is ample if*

(1) $d > \frac{95}{32}m$, *and*
(2) $d^2 \geqslant \left(\frac{3r+40}{3r+39}\right) rm^2$.

Proof. As in the proof of Theorem 2.1 we use the Nakai-Moishezon criterion. Let $C = eH - \sum_{i=1}^r n_i E_i$ be an irreducible, reduced curve on X. We consider different cases.

Case 1: First suppose that $0 \leqslant n_i \leqslant 11$ for all i.

If $C^2 \geqslant 0$, then Lemma 2.10 implies that $L \cdot C > 0$.

If $C^2 < 0$, then C is a (-1)-curve. See Remark 2.16. By Lemma 2.14, we have $e \leqslant 32$. So $96e - 95e \leqslant 32 \Rightarrow 3e - 1 \leqslant \frac{95}{32}e$. Further, since $K_X \cdot C = -1$, we get $\sum_{i=1}^r n_i = 3e - 1$. It follows now that $L \cdot C > 0$. For, by hypothesis (1), $de > \frac{95}{32}me \geqslant m(3e-1) = m\sum_{i=1}^r n_i$.

We remark that we do have several exceptional curves with $e = 32$. For instance, consider the linear system $D = 32H - 15E_1 - 10\sum_{i=2}^9 E_i$. It is possible to reduce D to E_9 by applying elements of W_9 (see Discussion 2.13). As we noted in the proof of Lemma 2.14, Nagata showed that exceptional curves form a single orbit under the action of W_9. So it follows that D is a (-1)-curve.

Case 2: We assume now that $n_1 \geqslant 12$. Write $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. Without loss of generality, we assume that $n_r \neq 0$.

Case 2(i): Suppose that $n_r = 1$.

By [X1, Lemma 1], $e^2 \geqslant n_1^2 + n_2^2 + \ldots + n_{r-1}^2$. Thus $C^2 = e^2 - n_1^2 - n_2^2 - \ldots n_{r-1}^2 - 1 \geqslant -1$. If $C^2 \geqslant 0$, then we are done by Lemma 2.10. So suppose that $C^2 = -1$.

Let $a = \sum_{i=1}^r n_i^2$ and $b = \sum_{i=1}^r n_i$. Then $e^2 = a - 1$.

We claim that $ra - b^2 \geqslant r$. Indeed, we have $ra - b^2 = \sum_{i,j}(n_i - n_j)^2$. Since we have $n_1 \geqslant 12$ and $n_r = 1$, the number of non-zero terms in the sum $\sum_{i,j}(n_i - n_j)^2$ is at least $r-1$. In fact, the number of non-zero terms is at least r unless the $(n_1,\ldots,n_r) = (12,\ldots,12,1)$ or $(n_1,\ldots,n_r) = (12,1,\ldots,1)$. In both cases it is clear that the sum $\sum_{i,j}(n_i - n_j)^2$ is at least r.

Now $ra - r \geqslant b^2 \Rightarrow a - 1 \geqslant \frac{b^2}{r}$. Thus $d^2e^2 > (a-1)rm^2 \geqslant r\frac{b^2}{r}m^2 = b^2m^2$ and $de > bm$.

Case 2(ii): Suppose that $n_r = 2$.

By [dF, Theorem 2.5], any irreducible, reduced curve which passes through one of the points p_i with multiplicity 2 is a (-1)-curve if it has negative self-intersection. Thus if $C^2 < 0$ then C is a (-1)-curve. In particular, $C^2 = e^2 - \sum_{i=1}^{r} n_i^2 = -1$. Exactly as in **Case 2(i)** we conclude $L \cdot C > 0$. If $C^2 \geqslant 0$, we are done by Lemma 2.10.

Case 2(iii): $n_r \geqslant 3$.

Suppose first that $r \leqslant 8$. According to hypothesis (1), $d > \left(\frac{95}{32}\right) m = 2.96875m$. In the notation of Proposition 2.11, $\lambda_8 = 2.966$ and $\lambda_r \leqslant \lambda_8$ for $r \leqslant 8$. So L is ample by Proposition 2.11.

When $r \geqslant 9$, we use Lemma 2.17. We have, using (2.1),

$$e^2 \geqslant \sum_{i=1}^{r} n_i^2 - n_r > \frac{3r+40}{r(3r+39)} \left(\sum_{i=1}^{r} n_i \right)^2$$
$$\implies d^2 e^2 > m^2 \left(\sum_{i=1}^{r} n_i \right)^2 \implies de > m \left(\sum_{i=1}^{r} n_i \right) \implies L \cdot C > 0.$$

We conclude that L is ample. □

Remark 2.19. For $r < 9$, we can improve hypothesis (1) in Theorem 2.18. We needed the condition $d > \frac{95}{32} m$ in order to ensure that L to meet all the exceptional curves $eH - \sum_{i=1}^{r} n_i E_i$ with $0 \leqslant n_i \leqslant 11$ positively. For $r \leqslant 8$, we have fewer such exceptional curves. See the proof of Theorem 2.8. As an illustration, when $r = 8$, a modification of Theorem 2.18 along these lines requires only $d > 2.66m$ as hypothesis (1), while hypothesis (2) is unchanged.

Example 2.20. We re-visit Example 2.12: $r = 8, m = 60$. We already found that L_{178} is ample using Theorem 2.1. We can conclude now that L_{172} is ample using Theorem 2.18 and Remark 2.19. Note that $60\sqrt{\frac{(8)(64)}{63}} = 171.04$ and $172 > (2.66)(60) = 159.6$.

In fact, it turns out that L_{171} is ample. Let $F = 17H - 6\sum_{i=1}^{8} E_i$. Then $L_{170} = 10F$. There is an element $w \in W_8$ such that $wF = H$. See Discussion 2.13. This is easy to see by applying the linear map $\gamma \in W_8$ successively to F and permuting E_i so that their coefficients are non-increasing. Since W_8 preserves intersections and H is nef, F is nef as well. It now follows that $L_{171} = 10F + H$ is ample: let $C = dH - \sum_{i=1}^{8} n_i E_i$ be an irreducible, reduced curve. Since H is nef and $H \cdot C = d$, it follows that $d \geqslant 0$. If $d > 0$, then $L_{171} \cdot C \geqslant H \cdot C > 0$. Finally suppose that $d = 0$. If $n_i < 0$ for some i then $C \cdot E_i = n_i < 0$. This implies that $C = E_i$ and so $L_{171} \cdot C > 0$. On the other hand, if $n_i \geqslant 0$ for all i then C can not be effective as it is the negative of an effective curve.

Example 2.21. We re-visit Example 2.5 to show that $L_7 = 7H - 3E_1 - \sum_{i=2}^{10} 2E_i - \sum_{i=11}^{12} E_i$ is ample if the SHGH Conjecture is true. To prove this, first consider an irreducible and reduced curve C on X with $C^2 < 0$. By the SHGH Conjecture, C is a (-1)-curve. Then as we noted in the proof of Lemma 2.14, there exists $w \in W_{12}$ such that $C = wE_{12}$. In fact, one can write $C = \sum_{i=0}^{r-1} a_i s_i + E_{12}$ where s_i are simple roots and $a_i \geqslant 0$; see [L1, Proposition 1.11]. It is easy to see that $L \cdot s_i \geqslant 0$ and hence we have $L \cdot C \geqslant L \cdot E_{12} > 0$.

Now let C be an irreducible, reduced curve with $C^2 \geqslant 0$. Since $L_7^2 > 0$, a sufficiently large multiple of L_7 is effective by Riemann-Roch. Thus $L_7 \cdot C \geqslant 0$. If $L_7 \cdot C = 0$, then Hodge Index Theorem gives $C^2 < 0$ contradicting the assumption on C.

Contrary to the situation when $r < 9$, the necessary condition $L^2 > 0$ is also conjectured to be sufficient for ampleness in the uniform case when $r \geqslant 9$. Recall the well-known Nagata Conjecture [N1]:

Nagata Conjecture. *Let $p_1, \dots, p_r$ be general points of $\mathbb{P}^2$ with $r \geqslant 9$. Let $n_1, \dots, n_r$ be non-negative integers. Let C be a curve of degree e in $\mathbb{P}^2$ passing through p_i with multiplicity at least n_i. Then*

$$e \geqslant \frac{1}{\sqrt{r}} \sum_{i=1}^{r} n_i,$$

and the inequality is strict for $r \geqslant 10$.

The Nagata Conjecture clearly implies (when $r \geqslant 9$) that $L = dH - m\sum_{i=1}^r E_i$ is ample if $L^2 > 0$. Indeed, since $L^2 > 0$ we have $d^2 > rm^2$. Now if C is an irreducible and reduced curve on X in the linear system of $eH - \sum_{i=1}^r n_i E_i$, then $e \geqslant \frac{1}{\sqrt{r}} \sum_{i=1}^r n_i$ by the Nagata Conjecture. Hence $de > m \sum_{i=1}^r n_i$ and $L \cdot C > 0$.

The conjecture is open except when r is a square in which case Nagata proved it. Some recent work on this conjecture can be found in [H6, H8, Ro]. For a nice survey, see [SS].

Example 2.22. Let $L_{d,r,m} = dH - m\sum_{i=1}^r E_i$.

(1) First consider $r = m = 10$. Then $L^2_{d,10,10} = d^2 - 1000$. The Nagata Conjecture predicts that $L_{32,10,10}$ is ample. Using Theorem 2.18 we can verify this. Since $10\sqrt{\frac{(10)(70)}{69}} = 31.85$, $L_{32,10,10}$ is ample by Theorem 2.18.
(2) Let $r = 10, m = 30$. Then $L^2_{d,10,30} = d^2 - 9000$. The Nagata Conjecture predicts that $L_{95,10,30}$ is ample. Using Theorem 2.18 we see that $L_{96,10,30}$ is ample since $30\sqrt{\frac{(10)(70)}{69}} = 95.55$.
(3) Let $r = 30, m = 10$. Then $L^2_{d,30,10} = d^2 - 3000$. The Nagata Conjecture predicts that $L_{55,10,30}$ is ample. Using Theorem 2.18 we can indeed verify that $L_{55,30,10}$ is ample since $10\sqrt{\frac{(30)(130)}{129}} = 54.98$.

[ST3, Theorem 3] gives similar conditions for ampleness in the uniform case. It says that $L_{d,r,m}$ is ample if $d \geqslant 3m+1$ and $d^2 \geqslant (r+1)m^2$. In Example 2.22, it gives ampleness of $L_{34,10,10}$, $L_{100,10,30}$, and $L_{56,30,10}$ in (1),(2) and (3) respectively. Our bound will always be at least as good as this because $\frac{(3r+40)r}{3r+39} < r+1$.

Remark 2.23. When $m = 1$, [K, Corollary] and [X2, Theorem] prove the optimal result about ampleness predicted by the Nagata Conjecture: L is ample if and only if $L^2 > 0$. Our Theorem 2.18, as well as [ST3, Theorem 3], recover this result.

3. Global generation and very ampleness

In this section, using similar methods as above, we obtain conditions for global generation and very ampleness of L applying Reider's criterion [Re].

Let X be a smooth surface and let N be a nef line bundle on X. Reider's theorem says that if $N^2 \geqslant 5$ and $K_X + N$ fails to be globally generated, there exists an effective divisor D such that

$D \cdot N = 0, D^2 = -1$, or

$D \cdot N = 1, D^2 = 0$.

Similarly, if $N^2 \geqslant 10$ and $K_X + N$ fails to be very ample, there exists an effective divisor D such that

$D \cdot N = 0, D^2 = -1 \text{ or } -2$, or

$D \cdot N = 1, D^2 = 0 \text{ or } -1$, or

$D \cdot N = 2, D^2 = 0$.

We apply Reider's theorem to $L = dH - \sum_{i=1}^{r} m_i E_i$ to obtain conditions for global generation and very ampleness.

First we consider conditions for global generation.

Theorem 3.1. *Let $p_1, p_2, \ldots, p_r$ be general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ not passing through any of the points p_i and by $E_1, \ldots, E_r$ the exceptional divisors. Let $L = dH - \sum_{i=1}^{r} m_i E_i$. Suppose that $r \geqslant 5$ and $m_i \geqslant 2$ for all i. Then L is globally generated if the following conditions hold:*

(1) $d + 3 > m_1 + m_2 + 2$,
(2) $2(d + 3) > m_1 + \ldots + m_5 + 5$,
(3) $3(d + 3) > 2m_1 + m_2 + \ldots + m_7 + 8$, *and*
(4) $(d + 3)^2 \geqslant \frac{s+3}{s+2} \sum_{i=1}^{s} (m_i + 1)^2$, *for* $2 \leqslant s \leqslant r$.

Proof. Let $N = (d + 3)H - \sum_{i=1}^{r}(m_i + 1)E_i$, so that $L = K_X + N$. Using our hypotheses we can apply Theorem 2.1 and conclude that N is ample. Further $N^2 = (d + 3)^2 - \sum_{i=1}^{r} (m_i + 1)^2 \geqslant \frac{1}{r+2} \sum_{i=1}^{r} (m_i + 1)^2$. Since $r \geqslant 5$ and $m_i \geqslant 2$ for all i, it follows that $N^2 \geqslant 5$. Thus we can apply Reider's theorem. Note that since N is ample, there is no effective divisor D such that $D \cdot N = 0$. Hence if L is not globally generated there exists an effective divisor D such that $D^2 = 0$ and $D \cdot N = 1$.

Suppose $D = eH - \sum_{i=1}^{r} n_i E_i$ with $e > 0$ and n_i non-negative integers. Without loss of generality, assume that $n_i > 0$ for all i. Then we have $e^2 = \sum_{i=1}^{r} n_i^2$ and $(d + 3)e = \sum_{i=1}^{r}(m_i+1)n_i+1$. For simplicity, let $a = \sum_{i=1}^{r} (m_i + 1)^2$, $b = \sum_{i=1}^{r} n_i^2$ and $c = \sum_{i=1}^{r}(m_i+1)n_i$.

We show that $(d + 3)e > c + 1$, which contradicts $D \cdot N = 1$.

We have $(d+3)^2e^2 > \frac{r+3}{r+2}ab$. We proceed as in the proof of Lemma 2.3 to show that $\frac{r+3}{r+2}ab \geqslant (c+1)^2$. This inequality is equivalent to $ab + (r+2)(ab-c^2) \geqslant (r+2)(2c+1)$. Since $ab - c^2 \geqslant 0$, this follows if $ab \geqslant (r+2)(2c+1)$. Using the inequality $ab \geqslant c^2$ again, it suffices to prove that $c^2 \geqslant (r+2)(2c+1)$. It is clear that the least value of $c^2-(r+2)(2c+1)$ is attained when $m_i + 1$ and n_i take the least values allowed. By hypothesis, $m_i \geqslant 2$ and $n_i \geqslant 1$. So the required inequality in this case is $9r^2 \geqslant (r+2)(6r+1)$. This holds for $r \geqslant 5$. □

Example 3.2. As in Example 2.4, consider $L_d = dH - 3E_1 - \sum_{i=2}^{8} 2E_i - \sum_{i=9}^{12} E_i$. Then the smallest d which satisfies the hypotheses (1)-(4) of Theorem 3.1 is $d = 8$: $(8+3)^2 > \frac{15}{14}(99) = 106.07$. It is clear that $d = 8$ satisfies the other hypotheses in Theorem 3.1. So L_8 is base point free. We saw that L_7 was ample in Example 2.4.

Example 3.3. As in Example 2.5, consider $L_d = dH - 3E_1 - \sum_{i=2}^{10} 2E_i - \sum_{i=11}^{12} E_i$. Then the smallest d which satisfies the hypotheses (1)-(4) of Theorem 3.1 is $d = 8$. By hypothesis (4), we require $(d+3)^2 \geqslant \frac{15}{14}(105) = 112.5$. It is easy to see that $d = 8$ satisfies all the other hypotheses. So $L_8 = 8H - 3E_1 - \sum_{i=2}^{10} 2E_i - \sum_{i=11}^{12} E_i$ is base point free. As we saw in Example 2.5, L_8 is also ample.

Example 3.4. As in Example 2.6, consider $L_d = dH - 10\sum_{i=1}^{5} E_i$. By Theorem 3.1, L_{25} is globally generated. It is easy to see that the required conditions are satisfied.

Our next result gives better bounds for global generation in the uniform case when $m \geqslant 6$.

Theorem 3.5. *Let $p_1, p_2, \ldots, p_r$ be general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ not passing through any of the points p_i and by $E_1, \ldots, E_r$ the exceptional divisors. Let $L = dH - m\sum_{i=1}^{r} E_i$. Suppose that $r \geqslant 2$ and $m \geqslant 6$. Then L is globally generated if the following conditions hold:*

(1) $d \geqslant \frac{95}{32}m$, *and*
(2) $(d+3)^2 \geqslant \left(\frac{3r+5}{3r+4}\right)r(m+1)^2$.

Proof. As in the proof of Theorem 3.1, let $N = (d+3)H - (m+1)\sum_{i=1}^{r} E_i$, so that $L = K_X + N$. Using our hypotheses we can apply Theorem 2.18 and conclude that N is ample. Note that $\frac{3r+5}{3r+4} > \frac{3r+40}{3r+29}$. Further $N^2 = (d+3)^2 - r(m+1)^2 \geqslant \frac{r(m+1)^2}{3r+4}$. Since $m \geqslant 6$, it is clear that $N^2 \geqslant 5$. We can thus apply Reider's theorem. If L is not globally generated there exists an effective divisor D such that $D^2 = 0$ and $D \cdot N = 1$.

Suppose $D = eH - \sum_{i=1}^{r} n_iE_i$ with $e > 0$ and n_i non-negative integers. Without loss of generality, assume that $n_i > 0$ for all i. Write $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r > 0$. Let $b = \sum_{i=1}^{r} n_i^2$ and $c = \sum_{i=1}^{r} n_i$. Then $e^2 = b$.

By hypothesis, $(d+3)^2e^2 \geqslant \left(\frac{3r+5}{3r+4}\right)(m+1)^2rb$. Note that $D \cdot N = (d+3)e - (m+1)c$. We claim now that $\left(\frac{3r+5}{3r+4}\right)(m+1)^2rb > \left((m+1)c+1\right)^2$, which contradicts $D \cdot N = 1$.

The required inequality is equivalent to

$$(3r+5)(m+1)^2rb > (3r+4)\big((m+1)^2c^2+2(m+1)c+1\big)$$

$$\Leftrightarrow (m+1)^2rb + (3r+4)(m+1)^2(rb-c^2) > (3r+4)(2(m+1)c+1).$$

Since $rb - c^2 \geqslant 0$, it suffices to show that

$$(m+1)^2rb > (3r+4)(2(m+1)c+1) = 6(m+1)rc + 8(m+1)c + 3r + 4. \tag{3.1}$$

First suppose that $n_1 = 1$. Then we have $e^2 = r$. By a dimension count, we see that for $e \geqslant 4$, such curves don't exist. So only such curves are $D = H - E_1$, $D = 2H - E_1 - \ldots - E_4$ and $D = 3H - E_1 - \ldots - E_9$. Clearly $N \cdot (H - E_1) > 1$. So assume that $e \geqslant 2$. We have $N^2 > 0$ by hypothesis (2). So $(d+3)^2 > (m+1)^2r \Longrightarrow d+3 > (m+1)e \Longrightarrow (d+3)-(m+1)e \geqslant 1 \Longrightarrow (d+3)e - (m+1)r \geqslant e \geqslant 2$. Hence $N \cdot D > 1$.

Now suppose that $n_1 \geqslant 2$. We re-write (3.1) as follows.

$$b > c\left(\frac{6}{m+1} + \frac{8}{r(m+1)}\right) + \frac{3r+4}{(m+1)^2r}. \tag{3.2}$$

Note that $b = \sum n_i^2$ grows faster than $c = \sum n_i$. For fixed r and m, if (3.2) holds for $(n_1, n_2, \ldots, n_r)$ then it holds for $(n_1, \ldots, n_i + 1, \ldots, n_r)$. Thus it suffices to check (3.2) for $n_1 = 2, n_2 = \ldots = n_r = 1$. In this case, $b = r+3$ and $c = r+1$. So (3.2) is equivalent to

$$(r+3) > (r+1)\left(\frac{6}{m+1} + \frac{8}{r(m+1)}\right) + \frac{3r+4}{(m+1)^2r} = \frac{r+1}{m+1}\left(6 + \frac{8}{r}\right) + \frac{3r+4}{(m+1)^2r}.$$

It is easy to see that this holds when $m \geqslant 6$ and $r \geqslant 2$. □

Example 3.6. As in Example 2.22, let $L_{d,r,m} = dH - m\sum_{i=1}^{r} E_i$.

(1) $r = m = 10$. By Theorem 3.5, $L_{33,10,10}$ is base point free since $11\sqrt{\frac{(10)(35)}{34}} = 35.29$. Recall that $L_{32,10,10}$ is ample by Theorem 2.18.
(2) $r = 10, m = 30$. Since $31\sqrt{\frac{(10)(35)}{34}} = 99.46$, $L_{97,10,30}$ is base point free. Recall that $L_{96,10,30}$ is ample.
(3) Let $r = 30, m = 10$. $L_{58,30,10}$ is base point free since $11\sqrt{\frac{(30)(95)}{94}} = 60.56$. Recall that $L_{55,30,10}$ is ample.

[ST3, Theorem 4] gives similar conditions for global generation in the uniform case. It says that $L_{d,r,m}$ is globally generated if $d \geqslant 3m+1$ and $(d+3)^2 \geqslant (r+1)(m+1)^2$. In this example, it gives global generation of $L_{34,10,10}$, $L_{100,10,30}$, and $L_{59,30,10}$. Our bound will always be at least as good as this because $\frac{(3r+5)r}{3r+4} < r+1$.

We now give conditions for very ampleness of a line bundle in the uniform case.

Theorem 3.7. *Let $p_1, p_2, \ldots, p_r$ be general points of $\mathbb{P}^2$. Consider the blow up $X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$. We denote by H the pull-back of a line in $\mathbb{P}^2$ not passing through any of the points p_i and by $E_1, \ldots, E_r$ the exceptional divisors. Let $L = dH - m\sum_{i=1}^{r} E_i$. Suppose that $r \geqslant 3$ and $m \geqslant 4$. Then L is very ample if the following conditions hold:*

(1) $d \geqslant 3m$, *and*

(2) $(d+3)^2 \geqslant \left(\frac{r+3}{r+2}\right) r(m+1)^2$.

Proof. Let $N = (d+3)H - (m+1)\sum_{i=1}^r E_i$. Then N is ample by Theorem 2.18. Further, using hypothesis (2), we get $N^2 = (d+3)^2 - r(m+1)^2 \geqslant \frac{r}{r+2}(m+1)^2$. Since $r \geqslant 3$ and $m \geqslant 4$, we obtain $N^2 \geqslant 10$. So we can apply Reider's theorem.

Suppose that L is not very ample. By Reider's theorem there exists an effective divisor D such that

$D \cdot N = 1, D^2 = 0 \text{ or } -1$, or

$D \cdot N = 2, D^2 = 0$.

We rule out the case $D \cdot N = 1, D^2 = 0$ exactly as in the proof of Theorem 3.5.

Let D be an effective divisor such that $D \cdot N = 2, D^2 = 0$. As in the proof of Theorem 3.5, write $D = eH - \sum_{i=1}^r n_i E_i$ with $e > 0$ and n_i non-negative integers. Without loss of generality, assume that $n_i > 0$ for all i. Write $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r > 0$. Let $b = \sum_{i=1}^r n_i^2$ and $c = \sum_{i=1}^r n_i$. Then $e^2 = b$.

By hypothesis, $(d+3)^2 e^2 \geqslant \left(\frac{r+3}{r+2}\right)(m+1)^2 rb$. Note that $D \cdot N = (d+3)e - (m+1)c$. We claim now that $\left(\frac{r+3}{r+2}\right)(m+1)^2 rb > ((m+1)c+2)^2$, which contradicts $D \cdot N = 2$.

The required inequality is equivalent to

$(r+3)(m+1)^2 rb > (r+2)\big((m+1)^2c^2 + 4(m+1)c + 4\big)$

$\Leftrightarrow (m+1)^2 rb + (r+2)(m+1)^2(rb - c^2) > (r+2)(4(m+1)c+4)$.

Since $rb - c^2 \geqslant 0$, it suffices to show that

$$(m+1)^2 rb > (r+2)(4(m+1)c+4) = 4(m+1)rc + 8(m+1)c + 4r + 8. \tag{3.3}$$

If $n_1 = 1$ we argue exactly as in the proof of Theorem 3.5. So suppose that $n_1 \geqslant 2$.

We re-write (3.3) as follows.

$$b > c\left(\frac{4}{m+1} + \frac{8}{r(m+1)}\right) + \frac{4r+8}{(m+1)^2 r}. \tag{3.4}$$

Note that $b = \sum n_i^2$ grows faster than $c = \sum n_i$. For fixed r and m, if (3.4) holds for $(n_1, n_2, \ldots, n_r)$ then it holds for $(n_1, \ldots, n_i + 1, \ldots, n_r)$. Thus it suffices to check (3.4) for $n_1 = 2, n_2 = \ldots = n_r = 1$. In this case, $b = r+3$ and $c = r+1$. So (3.4) is equivalent to

$$(r+3) > (r+1)\left(\frac{4}{m+1} + \frac{8}{r(m+1)}\right) + \frac{4r+8}{(m+1)^2 r} = \frac{r+1}{m+1}\left(4 + \frac{8}{r}\right) + \frac{4r+8}{(m+1)^2 r}.$$

It is easy to see that this holds when $m \geqslant 4$ and $r \geqslant 3$.

Now we consider the case $D \cdot N = 1, D^2 = -1$. As above, write $D = eH - \sum_{i=1}^r n_i E_i$ with $e > 0$ and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r > 0$.

Since $D \cdot N = 1$ and N is ample, we may assume that D is an irreducible and reduced curve.

Suppose that $n_1 \leqslant 11$. By [DJ, Theorem 34], any irreducible, reduced curve of negative self-intersection and with multiplicities of 11 or less at p_i is a (-1)-curve. So D is a (-1)-curve and hence $D \cdot K_X = -1$. This means that $3e = 1 + \sum_i n_i$. As $d \geqslant 3m$, we have $d+3 \geqslant 3(m+1)$ and this implies that $(d+3)e \geqslant (m+1)(1+\sum_i n_i) = (m+1)+(m+1)\sum_i n_i$, so $D \cdot N \geqslant m+1 > 1$ which contradicts the hypothesis that $D \cdot N = 1$.

Next suppose that $n_1 \geqslant 12$. By Lemma 3.9 below, $\frac{r(r+3)}{r+2}e^2 > \left(\sum_{i=1}^r n_i + \frac{1}{2}\right)^2$. Thus $e^2 > \frac{r+2}{r(r+3)}\left(\sum_{i=1}^r n_i + \frac{1}{2}\right)^2$. By hypothesis (2), $(d+3)^2 \geqslant \frac{r(r+3)}{r+2}(m+1)^2$.

So $(d+3)^2 e^2 > (m+1)^2\left(\sum_{i=1}^r n_i + \frac{1}{2}\right)^2$, giving $(d+3)e > (m+1)\left(\sum_{i=1}^r n_i + \frac{1}{2}\right)$. Then $(d+3)e - (m+1)\left(\sum_{i=1}^r n_i\right) > \frac{m+1}{2} \geqslant 2$, again contradicting the hypothesis that $D \cdot N = 1$.

We conclude that L is very ample. □

Theorem 3.7 gives conditions for very ampleness for a line bundle of the form $dH - m\sum_{i=1}^r E_i$ if $m \geqslant 4$ and $r \geqslant 3$. The case $m = 1$ is well-studied. In fact, a complete characterization is known in this case, see [GGP, Proposition 3.2].

Example 3.8. As in Examples 2.22 and 3.6, let $L_{d,r,m} = dH - m\sum_{i=1}^r E_i$.

(1) $r = m = 10$. By Theorem 3.7, $L_{34,10,10}$ is very ample since $11\sqrt{\frac{(10)(13)}{12}} = 36.20$. Recall that $L_{32,10,10}$ is ample and $L_{33,10,10}$ is base point free.
(2) $r = 10, m = 30$. Since $31\sqrt{\frac{(10)(13)}{12}} = 102.03$, $L_{100,10,30}$ is very ample. Recall that $L_{96,10,30}$ is ample and $L_{97,10,30}$ is base point free.
(3) Let $r = 30, m = 10$. $L_{59,30,10}$ is very ample since $11\sqrt{\frac{(30)(33)}{32}} = 61.18$. Recall that $L_{55,30,10}$ is ample and $L_{58,30,10}$ is base point free.

Finally we prove the following lemma which was used in the proof of Theorem 3.7.

Lemma 3.9. *Let $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r > 0$ be positive integers with $r \geqslant 2$ and $n_1 \geqslant 12$. Then $\frac{(r+3)r}{r+2}(n_1^2 + n_2^2 + \ldots + n_r^2 - 1) > \left(n_1 + n_2 + \ldots + n_r + \frac{1}{2}\right)^2$.*

Proof. Let $a = n_1^2 + n_2^2 + \ldots + n_r^2$, $b = n_1 + n_2 + \ldots + n_r$ and $c = \sum_{i<j}(n_i - n_j)^2$. Note that $ra - b^2 = c$.

We first consider the uniform case: $n_1 = n_2 = \ldots n_r = n$. Then $a = rn^2, b = rn$. So the desired inequality is $\frac{(r+3)r}{r+2}(rn^2 - 1) > r^2n^2 + rn + \frac{1}{4}$. It is equivalent to

$$\begin{aligned} r^3n^2 + 3r^2n^2 - r^2 - 3r &> r^3n^2 + 2r^2n^2 + r^2n + 2rn + r/4 + 1/2 \\ \Longleftrightarrow r^2(n^2 - n - 1) &> (2n + 3.25)r + 1/2 \end{aligned}$$

This clearly holds for $r \geqslant 2, n \geqslant 12$.

If all n_i are not all equal to each other, then $ra - b^2 = c \geqslant r - 1$. Substituting $ra = b^2 + c$, the desired inequality is $(r+3)(b^2 + c - r) > (r+2)(b^2 + b + 1/4)$. Since $c - r \geqslant -1$, it suffices to show that $(r+3)(b^2 - 1) > (r+2)(b^2 + b + 1/4)$. This is equivalent to

$$b^2 > b(r+2) + r + r/4 + 3.5. \tag{3.5}$$

For a fixed r, it is easy to see that if (3.5) holds for some $b = \sum_i^r n_i$, then it holds for $b+1$ also. Therefore it suffices to check (3.5) for the smallest value of b. This is attained when $(n_1, n_2, \ldots, n_r) = (12, 1, \ldots, 1)$. In this case, $b = r + 143$ and (3.5) holds in this case, as one can see by an easy calculation. $\square$

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