NEGATIVE CURVES ON SPECIAL RATIONAL SURFACES

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Abstract. We study negative curves on surfaces obtained by blowing up special configurations of points in $\mathbb{P}^2$. Our main results concern the following configurations: very general points on an irreducible cubic, 3–torsion points on an elliptic curve and nine Fermat points. As a consequence of our analysis, we also show that the Bounded Negativity Conjecture holds for the surfaces we consider. The note contains also some problems for future attention.

1. Introduction

Negative curves on algebraic surfaces are an object of classical interest. One of the most prominent achievements of the Italian School of algebraic geometry was Castelnuovo's Contractibility Criterion.

Definition 1.1 (Negative curve). *We say that a reduced and irreducible curve C on a smooth projective surface is* negative, *if its self-intersection number C^2 is less than zero.*

Example 1.2 (Exceptional divisor, (-1)-curves). Let X be a smooth projective surface and let $P \in X$ be a closed point. Let $f : \mathrm{Bl}_P X \to X$ be the blow up of X at the point P. Then the exceptional divisor E of f (i.e., the set of points in $\mathrm{Bl}_P X$ mapped by f to P) is a negative curve. More precisely, E is rational and $E^2 = -1$. By a slight abuse of language we will call such curves simply (-1)–curves.

Castelnuovo's result asserts that the converse is also true, see [13, Theorem V.5.7] or [2, Theorem III.4.1].

Theorem 1.3 (Castelnuovo's Contractibility Criterion). *Let Y be a smooth projective surface defined over an algebraically closed field. If C is a rational curve with $C^2 = -1$, then there exists a smooth projective surface X and a projective morphism $f : Y \to X$ contracting C to a smooth point on X. In other words, Y is isomorphic to $\mathrm{Bl}_P X$ for some point $P \in X$.*

The above result plays a pivotal role in the Enriques-Kodaira classification of surfaces.

Of course, there are other situations in which negative curves on algebraic surfaces appear.

Example 1.4. Let C be a smooth curve of genus $g(C) \geq 2$. Then the diagonal $\Delta \subset C \times C$ is a negative curve as its self-intersection is $\Delta^2 = 2 - 2g$.

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It is quite curious that it is in general not known if for a general curve C, there are other negative curves on the surface $C \times C$, see [15]. It is in fact even more interesting, that there is a direct relation between this problem and the famous Nagata Conjecture. This was observed by Ciliberto and Kouvidakis [6].

There is also a connection between negative curves and the Nagata Conjecture on general blow ups of $\mathbb{P}^2$. We recall the following conjecture about (-1)-curves which in fact implies the Nagata Conjecture; see [5, Lemma 2.4].

Conjecture 1.5 (Weak SHGH Conjecture)**.** *Let $f : X \to \mathbb{P}^2$ be the blow up of the projective plane $\mathbb{P}^2$ in general points $P_1, \ldots, P_s$. If $s \geq 10$, then the only negative curves on X are the (-1)-curves.*

On the other hand, it is well known that already a blow up of $\mathbb{P}^2$ in 9 general points carries *infinitely* many (-1)–curves.

One of the central and widely open problems concerning negative curves on algebraic surfaces asks whether on a fixed surface negativity is bounded. More precisely, we have the following conjecture (BNC in short). See [3] for an extended introduction to this problem.

Conjecture 1.6 (Bounded Negativity Conjecture)**.** *Let X be a smooth projective surface. Then there exists a number τ such that*

$$C^2 \geq \tau$$

for any reduced and irreducible curve $C \subset X$.

If the Conjecture holds on a surface X, then we denote by $b(X)$ the largest number τ such that the Conjecture holds. It is known (see [3, Proposition 5.1]) that if the negativity of reduced and irreducible curves is bounded below, then the negativity of all reduced curves is also bounded below.

Conjecture 1.6 is known to fail in the positive characteristic; see [9, 3]. In fact Example 1.4 combined with the action of the Frobenius morphism provides a counterexample. In characteristic zero, Conjecture 1.6 is open in general. It is easy to prove BNC in some cases; see Remark 3.11 for an easy argument when the anti-canonical divisor of X is nef. However, in many other cases the conjecture is open. In particular the following question is open and answering it may lead to a better understanding of Conjecture 1.6.

Question 1.7. Let X, Y be smooth projective surfaces and suppose that X and Y are birational and Conjecture 1.6 holds for X. Then does Conjecture 1.6 hold for Y also?

This is not known even in the simplest case, when one of surfaces is $\mathbb{P}^2$ (where Conjecture 1.6 obviously holds) and the other is a blow up of $\mathbb{P}^2$. If we blow up general points, then this is governed by Conjecture 1.5. The question is of interest also for special configurations of points in $\mathbb{P}^2$ and we focus our research here on such configurations. More concretely, we consider some examples of such special rational surfaces and list all negative curves on them. In particular, we study blow ups of $\mathbb{P}^2$ at certain points which lie on elliptic curves. Our main results classify negative curves on such surfaces; see Theorems 2.1, 3.3 and 3.7. As a consequence, we show that Conjecture 1.6 holds for such surfaces. This recovers some existing results of Harbourne and Miranda [12], [11]. Additionally we compute values of the number $b(X)$ on such surfaces.

2. Very general points on an irreducible cubic

To put our results in Section 3 into perspective, we recall results on negative curves on blow ups of $\mathbb{P}^2$ at s very general points on an plane curve of degree 3. Geometry of such surfaces was studied by Harbourne in [10].

Theorem 2.1 (Points on a cubic curve)**.** *Let D be an irreducible and reduced plane cubic and let $P_1, \ldots, P_s$ be smooth points on D. Let $f : X \longrightarrow \mathbb{P}^2$ be the blow up at $P_1, \ldots, P_s$. If $C \subset X$ is any reduced and irreducible curve such that $C^2 < 0$, then*

a) *C is the proper transform of D, or*
b) *C is a (-1)-curve, or*
c) *C is a (-2)-curve.*

Moreover, if the points $P_1, \ldots, P_s$ are very general, then only cases a) and b) are possible.

Proof. The first part of Theorem follows from [11, Remark III.13] and also from our Remark 3.11. The "moreover" part follows from the following abstract argument. A negative curve on X is either a component of $-K_X$, or a (-1)-curve or a (-2)-curve. But a (-2)-curve is in $\ker(\mathrm{Pic}(X) \to \mathrm{Pic}^0(-K_X))$, which is 0 for very general points, so there are no (-2)-curves. □

Corollary 2.2. *Let X be a surface as in Theorem 2.1 with $s > 0$ very general points. Then Conjecture 1.6 holds for X and we have*

$$b(X) = \min\{-1,\ 9 - s\}.$$

3. Special points on a smooth cubic

In this section, we consider blow ups of $\mathbb{P}^2$ at 3-torsion points of an elliptic curve as well as the points of intersection of the Fermat arrangement of lines. In order to consider these two cases, we deal first with the following numerical lemma which seems quite interesting in its own right.

Lemma 3.1. *Let $m_1, \ldots, m_9$ be nonnegative real numbers satisfying the following 12 inequalities:*

$$m_1 + m_2 + m_3 \leq 1, \tag{3.1}$$
$$m_4 + m_5 + m_6 \leq 1, \tag{3.2}$$
$$m_7 + m_8 + m_9 \leq 1, \tag{3.3}$$
$$m_1 + m_4 + m_7 \leq 1, \tag{3.4}$$
$$m_2 + m_5 + m_8 \leq 1, \tag{3.5}$$
$$m_3 + m_6 + m_9 \leq 1, \tag{3.6}$$
$$m_1 + m_5 + m_9 \leq 1, \tag{3.7}$$
$$m_2 + m_6 + m_7 \leq 1, \tag{3.8}$$
$$m_3 + m_4 + m_8 \leq 1, \tag{3.9}$$
$$m_1 + m_6 + m_8 \leq 1, \tag{3.10}$$
$$m_2 + m_4 + m_9 \leq 1, \tag{3.11}$$
$$m_3 + m_5 + m_7 \leq 1. \tag{3.12}$$

Then $m_1^2 + \cdots + m_9^2 \leq 1$.

Proof. Assume that the biggest number among $m_1, \ldots, m_9$ is $m_1 = 1 - m$ for some $0 \leq m \leq 1$.

Consider the following four pairs of numbers

$$p_1 = (m_2, m_3),\ p_2 = (m_4, m_7),\ p_3 = (m_9, m_5),\ p_4 = (m_6, m_8).$$

These are pairs such that together with m_1 they occur in one of the 12 inequalities. In each pair one of the numbers is greater or equal than the other. Let us call this bigger number a *giant*. A simple check shows that there are always three pairs, such that their giants are subject to one of the 12 inequalities in the Lemma.

Without loss of generality, let p_1, p_2, p_3 be such pairs. Also without loss of generality, let m_2, m_4 and m_9 be the giants. Thus $m_2 + m_4 + m_9 \leq 1$. Assume that also m_6 is a giant.

Inequality $m_2 + m_3 \leq m$ implies that

$$m_2^2 + m_3^2 = (m_2 + m_3)^2 - 2m_2m_3 \leq m(m_2 + m_3) - 2m_2m_3.$$

Observe also that

$$(m_2 + m_3)^2 - 4m_2m_3 \leq m(m_2 - m_3).$$

Analogous inequalities hold for pairs p_2, p_3 and p_4. Therefore

$$\begin{gathered} m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 \leq \\ \leq m(m_2 + m_4 + m_9 + m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9 \leq \\ \leq m + \big[m(m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9\big]. \end{gathered}$$

But we have also

$$m_2^2+m_3^2+m_4^2+m_7^2+m_5^2+m_9^2=$$
$$=(m_2+m_3)^2+(m_4+m_7)^2+(m_5+m_9)^2-2m_2m_3-2m_4m_7-2m_5m_9=$$
$$=(m_2+m_3)^2-4m_2m_3+(m_4+m_7)^2-4m_4m_7+$$
$$+(m_5+m_9)^2-4m_5m_9+2m_2m_3+2m_4m_7+2m_5m_9\leq$$
$$\leq m(m_2-m_3)+m(m_4-m_7)+m(m_9-m_5)+2m_2m_3+2m_4m_7+2m_5m_9\leq$$
$$\leq m-\left[m(m_3+m_7+m_5)-2m_2m_3-2m_4m_7-2m_5m_9\right],$$

which obviously gives

$$m_2^2+m_3^2+m_4^2+m_7^2+m_5^2+m_9^2\leq m.$$

Since

$$m_6^2+m_8^2\leq m_6^2+m_6m_8\leq m_6(m_6+m_8)\leq(1-m)m,$$

we get that the sum of all nine squares is bounded by

$$(1-m)^2+m+(1-m)m=1.$$ □

If we think of numbers $m_1,\ldots,m_9$ as arranged in a 3×3 matrix

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix},$$

then the inequalities in the Lemma 3.1 are obtained considering the horizontal, vertical triples and the triples determined by the condition that there is exactly one element m_i in every column and every row of the matrix (so determined by permutation matrices). Bounding sums of only such triples allows us to bound the sum of squares of all entries in the matrix. It is natural to wonder, if this phenomena extends to higher dimensional matrices. One possible extension is formulated as the next question.

Problem 3.2. Let $M=(m_{ij})_{i,j=1\ldots k}$ be a matrix whose entries are non-negative real numbers. Assume that all the horizontal, vertical and permutational k-tuples of entries in the matrix M are bounded by 1. Is it true then that the sum of squares of all entries of M is also bounded by 1?

3.1. **Torsion points.** We now consider a blow up of $\mathbb{P}^2$ at 9 points which are torsion points of order 3 on an elliptic curve embedded as a smooth cubic.

Theorem 3.3 (3–torsion points on an elliptic curve)**.** *Let D be a smooth plane cubic and let $P_1,\ldots,P_9$ be the flexes of D. Let $f:X\to\mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $P_1,\ldots,P_9$. If C is a negative curve on X, then*

a) *C is the proper transform of a line passing through two (hence three) of the points $P_1,\ldots,P_9$, and $C^2=-2$ or*
b) *C is an exceptional divisor of f and $C^2=-1$.*

Proof. It is well known that there is a group law on D such that the flexes are 3–torsion points. Since any line passing through two of the torsion points automatically meets D in a third torsion point, there are altogether 12 such lines. The torsion points form a subgroup of D which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. We can pick this isomorphism so that

$$P_1 = (0,0),\ P_2 = (1,0),\ P_3 = (2,0),$$
$$P_4 = (0,1),\ P_5 = (1,1),\ P_6 = (2,1),$$
$$P_7 = (0,2),\ P_8 = (1,2),\ P_9 = (2,2).$$

This implies that the following triples of points are collinear (note that these are exactly triples of indices in inequalities from (3.2) to (3.10):

$$(P_1,P_2,P_3),\ (P_4,P_5,P_6),\ (P_7,P_8,P_9),\ (P_1,P_4,P_7),$$
$$(P_2,P_5,P_8),\ (P_3,P_6,P_9),\ (P_1,P_5,P_9),\ (P_2,P_6,P_7),$$
$$(P_3,P_4,P_8),\ (P_1,P_6,P_8),\ (P_2,P_4,P_9),\ (P_3,P_5,P_7).$$

Let C be a reduced and irreducible curve on X different from the exceptional divisors of f and the proper transforms of lines through the torsion points. Then C is of the form

$$C = dH - k_1E_1 - \ldots - k_9E_9,$$

where $E_1,\ldots,E_9$ are the exceptional divisors of f and $k_1,\ldots,k_9 \geq 0$ and $d > 0$ is the degree of the image $f(C)$ in $\mathbb{P}^2$.

For $i = 1,\ldots,9$, let $m_i = \frac{k_i}{d}$. Since C is different from proper transforms of the 12 lines distinguished above, taking the intersection product of C with the 12 lines, and dividing by d, we obtain exactly the 12 inequalities in Lemma 3.1. The conclusion of Lemma 3.1 implies then that

$$C^2 = d^2 - \sum_{i=1}^{9} m_i^2 \geq 0,$$

which finishes our argument. □

Corollary 3.4. *For the surface X in Theorem 3.3 Conjecture 1.6 holds with*

$$b(X) = -2.$$

Remark 3.5. Theorem 3.3 fits in a more general setting of elliptic fibrations. Negative curves on surfaces X with $h^0(X, -mK_X) \geq 2$ for some $m \geq 2$ have been studied by Harbourne and Miranda in [12].

The observation in Remark 3.5 allows us to explain results of Theorem 3.3 from another point of view. Let $\widetilde{D}$ be the proper transform of D. Then, it is a member of the Hesse pencil, see [1], in particular the linear system $|\widetilde{D}|$ defines a morphism from X to $\mathbb{P}^1$. The components of reducible fibers are (-2) curves. There are 12 of them and they are proper transforms of lines passing through triples of blown-up points. The exceptional divisors over these points are the (-1) curves. These are sections of the fibration determined by $\widetilde{D}$.

Clearly Corollary 3.4 follows also from the adjunction and the fact that $-K_X$ is effective, see Remark 3.11. Of course, there is no reason to restrict to 3–torsion points.

Remark 3.6. With the same approach one can show that $m \geq 4$ the Bounded Negativity Conjecture holds on the blow ups of $\mathbb{P}^2$ at all the m–torsion points of an elliptic curve embedded as a smooth cubic and we have

$$b(X) = 9 - m^2.$$

3.2. **Fermat configuration of points.** The 9 points and 12 lines considered in subsection 3.1 form the famous Hesse arrangement of lines; see [14]. Any such arrangement is projectively equivalent to that obtained from the flex points of the Fermat cubic $x^3 + y^3 + z^3 = 0$ and the lines determined by their pairs. Explicitly in coordinates we have then

$$P_1 = (1 : \varepsilon : 0),\ P_2 = (1 : \varepsilon^2 : 0),\ P_3 = (1 : 1 : 0),$$

$$P_4 = (1 : 0 : \varepsilon),\ P_5 = (1 : 0 : \varepsilon^2),\ P_6 = (1 : 0 : 1),$$

$$P_7 = (0 : 1 : \varepsilon),\ P_8 = (0 : 1 : \varepsilon^2),\ P_9 = (0 : 1 : 1),$$

for the points and

$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 0, x + y + \varepsilon z = 0,\ x + y + \varepsilon^2 z = 0$$

$$x+\varepsilon y+z = 0,\ x+\varepsilon^2 y+z = 0,\ x+\varepsilon y+\varepsilon z = 0,\ x+\varepsilon y+\varepsilon^2 z = 0,\ x+\varepsilon^2 y+\varepsilon z = 0, x+\varepsilon^2 y+\varepsilon^2 z = 0,$$

for the lines, where ε is a primitive root of unity of order 3.

Passing to the dual plane, we obtain an arrangement of 9 lines defined by the linear factors of the Fermat polynomial

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0.$$

These lines intersect in triples in 12 points, which are dual to the lines of the Hesse arrangement. The resulting dual Hesse configuration has the type $(9_4, 12_3)$ and it belongs to a much bigger family of Fermat arrangements; see [17]. Figure 1 is an attempt to visualize this arrangement (which cannot be drawn in the real plane due to the famous Sylvester-Gallai Theorem; for instance, see [16]).

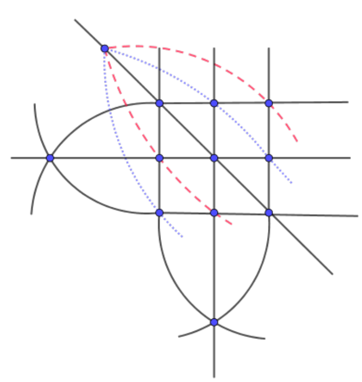

FIGURE 1. Fermat configuration of points

It is convenient to order the 9 intersection points in the affine part in the following way:

$$\begin{array}{lll} Q_1 = (\varepsilon : \varepsilon : 1), & Q_2 = (1 : \varepsilon : 1), & Q_3 = (\varepsilon^2 : \varepsilon : 1), \\ Q_4 = (\varepsilon : 1 : 1), & Q_5 = (1 : 1 : 1), & Q_6 = (\varepsilon^2 : 1 : 1), \\ Q_7 = (\varepsilon : \varepsilon^2 : 1), & Q_8 = (1 : \varepsilon^2 : 1), & Q_9 = (\varepsilon^2 : \varepsilon^2 : 1). \end{array}$$

With this notation established, we have the following result.

Theorem 3.7 (Fermat points)**.** *Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_9$. If C is a negative curve on X, then*

a) *C is the proper transform of a line passing through two or three of the points $Q_1, \ldots, Q_9$, or*
b) *C is a (-1)-curve.*

Proof. The proof of Theorem 3.3 works with very few adjustments.

Let us assume, to begin with, that C is a negative curve on X, distinct from the curves listed in the theorem. Then

$$C = dH - k_1E_1 - \ldots - k_9E_9,$$

for some $d > 0$ and $k_1, \ldots, k_9 \geq 0$. We can also assume that d is the smallest number for which such a negative curve exists. As before, we set

$$m_i = \frac{k_i}{d} \text{ for } i = 1, \ldots, 9.$$

Then the inequalities (3.1) to (3.9) follow from the fact that C intersects the 9 lines in the arrangement non-negatively.

If one of the remaining inequalities (3.10), (3.11) or (3.12) fails, then we perform a standard Cremona transformation based on the points involved in the failing inequality. For example, if (3.10) fails, we make Cremona based on points Q_1, Q_6 and Q_8. Note that these points are *not* collinear in the set-up of our Theorem. Since C is assumed not to be a line through any two of these points, its image C' under Cremona is a curve of strictly lower degree, negative on the blow up of $\mathbb{P}^2$ at the 9 points. The points $Q_1, \ldots, Q_9$ remain unchanged by the Cremona because, as already remarked, all dual Hesse arrangements are projectively equivalent. Then C' is again a negative curve on X of degree strictly lower than d, which contradicts our choice of C such that $C \cdot H$ is minimal.

Hence, we can assume that the inequalities (3.10), (3.11) and (3.12) are also satisfied. Then we conclude exactly as in the proof of Theorem 3.3. □

Remark 3.8. The surface X considered in Theorem 3.7 is a *non-extremal Jacobian rational elliptic surface* and contains infinitely many (-1)-curves. See [12] for more details.

In fact, we are in the position to identify all these (-1)-curves. Let $L(X,Y)$ denote the line determine two distinct points X and Y. Let

$$\mathcal{L} = \{U_1 = L(Q_1, Q_6), U_2 = L(Q_1, Q_8), U_3 = L(Q_6, Q_8), V_1 = L(Q_2, Q_4),$$
$$V_2 = L(Q_2, Q_9), V_3 = L(Q_4, Q_9), W_1 = L(Q_3, Q_5), W_2 = L(Q_3, Q_7), W_3 = L(Q_5, Q_7)\}$$

Cremona	deg	Q_1	Q_6	Q_8	Q_2	Q_4	Q_9	Q_3	Q_5	Q_7
	1	1	1	0	0	0	0	0	0	0
$\boldsymbol{\varphi_2}$	2	1	1	0	1	1	1	0	0	0
φ_3	4	1	1	0	1	1	1	2	2	2
$\boldsymbol{\varphi_1}$	6	3	3	2	1	1	1	2	2	2
φ_2	9	3	3	2	4	4	4	2	2	2
$\boldsymbol{\varphi_3}$	12	3	3	2	4	4	4	5	5	5

TABLE 1. A series of Cremona transformations

be the set of lines determined by pairs of points Q_i, Q_j with $1 \leq i < j \leq 9$ which contain only 2 points Q_k. These lines can grouped in three "triangles", which is indicated by the letters U, V and W used to labeling relevant triples. Vertices of these triangles determine standard Cremona transformations, which we denote by φ_1 for the U-triangle, i.e., points Q_1, Q_6, Q_8 and φ_2 and φ_3 for the V and W-triangles respectively.

Corollary 3.9. *Let $C \subset X$ be a (-1)-curve. Then either $C \in \mathcal{L}$ or there exists a positive integer $r \geq 1$ and a sequence of Cremona transformations $\varphi = \varphi_{i_r} \circ \ldots \circ \varphi_{i_1}$ with $i_1, \ldots, i_r \in \{1, 2, 3\}$ such that C is the image under φ of one of the lines in $\mathcal{L}$.*

Proof. The statement follows directly from the proof of Theorem 3.7. There is an interesting regularity in applying Cremona transformation, which we would like to present additionally. This is done best by the way of an example. Recall that there is the following general rule concerning changes of degree and multiplicities, when applying Cremona transformation. Let φ be the standard Cremona transformation based on a triangle F, G, H. Let C be a curve of degree d, different from the three lines $L(F, G)$, $L(F, H)$ and $L(G, H)$ passing through the points F, G, H with multiplicities m_F, m_G, m_H. Let $k = d - m_F - m_G - m_H$. Then the image curve $C' = \varphi(C)$ has degree $d + k$ and multiplicities $m_F + k$, $m_G + k$, $m_H + k$ in the base points of the reverse Cremona transformation. In Table 3.2 we present how the line $L(Q_1, Q_6)$ transforms under the sequence of Cremona transformations indicated in the first column. If it is possible to perform one of 2 Cremonas, we indicate it by writing the chosen one in boldface. Of course, it is always possible to choose the Cremona performed in the last step but as this leads to nothing new, we ignore this option. The diagram in Figure 2 indicates possible bifurcations at the places where one of two Cremona transformations can be performed. For simplicity, we put only degree of resulting (-1)-curves in the diagram. □

The diagram in Figure 2 seems quite interesting in its own right. There is a vertical symmetry, and it leads to a scheme of numbers indicated in Table 2, which has some reminiscences to the Pascal's triangle.

Problem 3.10. Investigate numerical properties of the Cremona hexal. For example, find a direct formula for the entry in line i and column j.

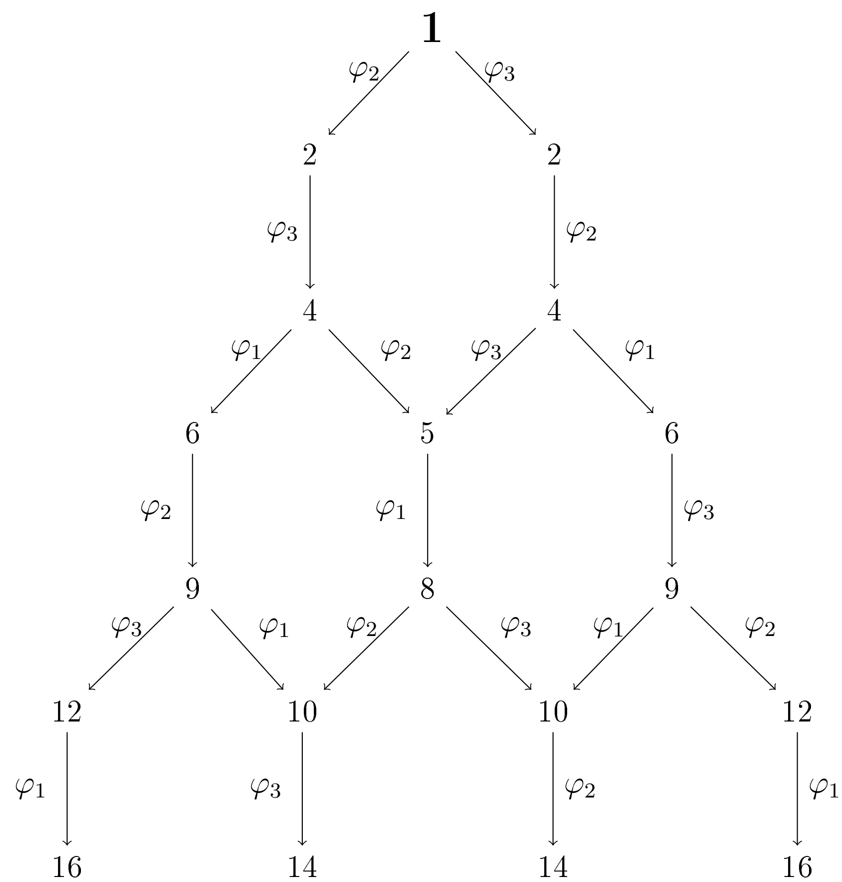

FIGURE 2. Bifurcations of Cremona transformations

			1			
		2		2		
		4		4		
	6		5		6	
	9		8		9	
12		10		10		12
16		14		14		16

TABLE 2. Cremona hexal

Remark 3.11. If we are interested only in the bounded negativity property on X, then there is a simple proof. Indeed, if $C \subset X$ is a reduced and irreducible curve, the genus formula gives

$$1 + \frac{C \cdot (C + K_X)}{2} \geq 0.$$

Now, since the anti-canonical divisor on the blow up of $\mathbb{P}^2$ in the 9 Fermat points is effective, we conclude that C is a component of $-K_X$ or

$$C^2 \geq -2 - CK_X \geq -2.$$

Having classified all the negative curves on the blow up of $\mathbb{P}^2$ at the 9 Fermat points, it is natural to wonder about the negative curves on blow ups of $\mathbb{P}^2$ arising from the other Fermat configurations. Note that the argument given in Remark 3.11 is no longer valid, since $-K_X$ is not nef or effective anymore. So it will be interesting to ask whether BNC holds for such surfaces. We pose the following problem.

Problem 3.12. For a positive integer m, let $Z(m)$ be the set of all points of the form

$$(1 : \varepsilon^{\alpha} : \varepsilon^{\beta}),$$

where ε is a primitive root of unity of order m and $1 \leq \alpha, \beta \leq m$. Let $f_m : X(m) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at all the points of $Z(m)$. Is the negativity bounded on $X(m)$? If so, what is the value of $b(X(m))$?

We end this note by the following remark which discusses bounded negativity for blow ups of $\mathbb{P}^2$ at 10 points.

Remark 3.13. Let X denote a blow up of $\mathbb{P}^2$ at 10 points. As mentioned before, if the blown up points are general, then Conjecture 1.5 predicts that the only negative curves on X are (-1)-curves. This is an open question. On the other hand, let us consider a couple of examples of special points.

Let X be obtained by blowing up the 10 nodes of an irreducible and reduced rational nodal sextic. Such surfaces are called *Coble surfaces* (these are smooth rational surfaces X such that $|-K_X| = \emptyset$, but $|-2K_X| \neq \emptyset$). Then it is known that BNC holds for X. In fact, we have $C^2 \geq -4$ for every irreducible and reduced curve $C \subset X$; see [4, Section 3.2].

Now let X be the blow up of 10 double points of intersection of 5 general lines in $\mathbb{P}^2$. Then $-K_X$ is a big divisor and by [18, Theorem 1], X is a *Mori dream space*. For such surfaces, the submonoid of the Picard group generated by the effective classes is finitely generated. Hence BNC holds for X ([9, Proposition I.2.5]).

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