Algebra IV, Mid-Semester Exam

September 25, 2017

Total Points: 60, Duration: 3 hours

1. (10 points) Let R be a PID.
   (a) State the structure theorem for finitely generated modules over R.
   (b) Define a projective module over R.
   (c) Show that a finitely generated module M over R is free if and only if it is projective.

2. (10 points) Find isomorphic direct product of cyclic groups, where V is the abelian group generated by x, y, z with the given relations.
   (a) $3x + 2y + 8z = 0$, $2x + 4z = 0$.
   (b) $4x - y + 5z = 0$, $14x + 7y + 7z = 0$.

3. (10 points) State true or false. Justify your answers.
   (a) Let R be a PID and let M be a finitely generated R-module. If M is generated by n elements and $N \subset M$ is a submodule then N can also be generated by n elements.
   (b) Let R be a ring. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated R-modules and M', M'' are free, then so is M.

4. (10 points) Let k be a field and let $R = k[x]/(x^2)$. We consider k as an R-module via the canonical quotient map $R \to k \cong R/(\bar{x})$. Construct an *infinite* free resolution for k as an R-module.

5. (10 points) Let R be an integral domain and let M be a finitely generated torsion-free R-module. Show that M is isomorphic to a submodule of a free R-module.

6. (10 points) Let R be a ring and let $a \in R$. Let $I := \{r \in R \mid ra = 0\}$ be the annihilator of a. Show that the ideal aR is a projective R-module if and only if I is a principal ideal generated by an idempotent $e \in R$ (i.e., e satisfies $e^2 = e$).