Topology, Mid-semester Exam
February 22, 2012

Do any **five** problems. Each problem is worth 20 points.

(1) (a) Show that a connected metric space having more than one point is uncountable.
(b) Let $\{A_n\}_{n\geq 1}$ be a sequence of connected subsets of a topological space X such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \geq 1$. Show that $\cup A_n$ is connected.

(2) Determine if the following pairs of spaces are homeomorphic. Justify your answers.
(a) $\mathbb{R}$ and $\mathbb{R}^n$, with $n > 1$;
(b) $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ and $\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ (in subspace topology);
(c) order topology and standard topology on $\mathbb{R}$;
(d) $\mathbb{R}_l$ and $\mathbb{R}$ under finite complement topology.

(3) True or False. Justify your answers.
(a) A space with a countable basis has a countable dense subset.
(b) A compact subset of any topological space is closed.
(c) A finite Hausdorff space is totally disconnected.
(d) Any ordered set is Hausdorff in the order topology.

(4) (a) Show that $[0,1] \subset \mathbb{R}_l$ is not limit point compact (in the subspace topology).
(b) Let (X,d) be a metric space and let $A \subset X$ be compact. Fix $x \in X \setminus A$. Show that there exist points $a_0, a_1 \in A$ such that $d(a_0, x) \leq d(a, x) \leq d(a_1, x)$ for all $a \in A$.

(5) Let X be a compact metric space and let $f : X \to X$ be an isometry. Show that f is a homeomorphism.

(6) Show that an arbitrary intersection of compact subsets of a **Hausdorff** space is compact. Give an example to show that the conclusion does not hold when X is not Hausdorff.

(7) Let $X = [0,1]^{[0,1]}$. Show that X (with product topology) is **not** sequentially compact by exhibiting a sequence in X which has no convergent subsequence.

Definitions

1. $\mathbb{R}_l$ is the set $\mathbb{R}$ with the lower limit topology: basic open sets are of the form $[a,b)$, for real numbers a, b.

2. Let (X,d) be a metric space. A continuous map $f : X \to X$ is called an **isometry** if $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.

3. A space is **limit point compact** if every infinite subset has a limit point.

4. A space X is **totally disconnected** if the only connected subsets of X are singletons.

5. Finite complement topology: U is open if it is $\emptyset$ or the complement of U is finite.