Mid term exam — Algebraic Surfaces — Oct 5, 2019

(i) A 'surface' refers to a nonsingular complex projective surface.
(ii) A 'negative curve' on a surface is an irr, red curve C s.t. $C^2 < 0$

1. Let X be a surface. Let D be a divisor on X s.t. $D.C \geq 0$ for all negative curves $C \subseteq X$. Show that D is nef.

   Hint: Use Zariski decomposition.

2. Let D be a divisor on a surface X. Suppose D has two decompositions: $D = P + N = P' + N'$ satisfying

   (a) $P.C \geq 0$, $P'.C \geq 0$ for all (-1)-curves $C \subseteq X$

   (b) $N = \Sigma a_i C_i$, $N' = \Sigma a_i' C_i'$, C_i, C_i' are (-1)-curves, $a_i \geq 0$, $a_i' \geq 0$ $\forall i$, $C_i . C_j = 0$ $\forall i \neq j$, $C_i' . C_j' = 0$ $\forall i, j$.

   (c) $PN = P'N' = 0$.

   Show that $P = P'$ and $N = N'$.

3. Prove the following stronger form of the Hodge index theorem. Let X be a surface; H, D are divisors on X s.t $H^2 > 0$, $HD = 0$. If D is not numerically trivial, show that $D^2 < 0$.

   Hint: Use the form of HIT we proved in class.

4. Let $X \to \mathbb{P}^2$ be a blowup at $r \geq 1$ collinear points in $\mathbb{P}^2$. Describe all negative curves on X.

5. Given a rational number $r > 0$, show that there exist a surface X, an ample line bundle L on X and $x \in X$ s.t $\varepsilon(X, L, x) = r$. Hint: Use Miranda's construction.

   You need not give the details of that construction.

6. A surface X is called a "coble surface" if $|-K_X| = \phi$ and $|-2K_X| \neq \phi$.

(a) Let $L_1, \dots, L_5 \subseteq \mathbb{P}^2$ be 5 lines meeting in 10 distinct points $x_1, \dots, x_{10} \in \mathbb{P}^2$ (this happens if we take 5 general lines, for example).

Let $X \to \mathbb{P}^2$ be a blowup at $x_1, \dots, x_{10}$. Show that X is a coble surface.

(b) Let $C \subseteq \mathbb{P}^2$ be an irr, red sextic with 10 points $x_1, \dots, x_{10}$ of multiplicity 2 (it is a fact that such a sextic exists). Let $X \to \mathbb{P}^2$ be the blowup of the ten double points of C.

(i) Show that X is a coble surface.

(ii) If $D \subseteq X$ is a negative rational curve, show that $D^2 = -1$, or $\begin{pmatrix} D^2 = -2, \\ D \cdot \tilde{C} = 0 \end{pmatrix}$, or $\begin{pmatrix} D^2 = -4, \\ D = \tilde{C} \end{pmatrix}$

Hint for (ii): Use adjunction formula.

7. A divisor D on a surface is called "big" if $\exists$ an ample divisor A and an effective divisor E s.t $mD = A + E$ for some positive integer m.

If D is big, show that the following set is finite:

$\mathrm{Null}(D) := \{ C \subseteq X \mid C \text{ is a negative curve s.t } D \cdot C = 0 \}$

8. If $-K_X$ is big and nef, show that the set of neg. curves C with $C^2 \leq -2$ is finite.