All varieties below are projective and defined over an algebraically closed field.

For problems (1) – (4): X is a nonsingular curve of genus g.

1) Let $D \in \operatorname{Div} X$ with $\deg D = 2g-2$.

a) Show that $h^0(D) = g-1$ or g;

b) Show that $h^0(D) = g \iff D = K_X$, where K_X is a canonical divisor of X.

2) a) Let E be an effective divisor on X; let $D \in \operatorname{Div}(X)$. Show that the following map of linear systems is well-defined and injective:

$$|D-E| \longrightarrow |D|$$
$$F \longmapsto F+E$$

Conclude that $h^0(D-E) \leq h^0(D)$.

b) Let $D \in \operatorname{Div}(X)$ be effective. Show that

$$h^0(D) - 1 \leq \deg D.$$

Hint: Use Riemann-Roch theorem, Serre duality and Part (a).

3) Let $D \in \operatorname{Div}(X)$, $P \in X$. Show that P is a base point of $D \iff h^0(D-P) = h^0(D)$.

4) Assume that $g \geq 2$. Show that K_X is base point free using the following fact:

FACT: Suppose $g > 0$. Then if P and Q are distinct points of X, then P and Q are <u>not</u> linearly equivalent.

Hint: Using FACT, show that $h^0(\mathcal{O}_X(P)) = 1$ for any point $P \in X$. Then use Riemann–Roch and problem (3).

For Problems (5)–(6), X is a nonsingular surface.

5) Let $H, D \in \mathrm{Div}(X)$ with H ample. Show that $D^2 \cdot H^2 \leq (D H)^2$.

Hint: Use the Hodge Index Theorem.

6) a) Let $C \subseteq X$ be a nonsingular curve contained in X. Assume that C is isomorphic to $\mathbb{P}^1$. If $C^2 = -1$, show that $K_X \cdot C = -1$.

Hint: Use the adjunction formula.

A curve C satisfying the above conditions is called "an exceptional curve of the first kind".

b) Assume that K_X is <u>NOT</u> effective (i.e., $h^0(K_X) = 0$). Let $C \subseteq X$ be an exceptional curve of the first kind. Show that $h^1(nC) \neq 0$ for $n \geq 2$.

Hint: Use Riemann–Roch, Serre duality.