SESHADRI CONSTANTS ON SURFACES

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1. Preliminaries

By a *surface*, we mean a projective nonsingular variety of dimension 2 over $\mathbb{C}$. A *curve* C on a surface X is an effective divisor. The group of divisors on X modulo linear equivalence is isomorphic to isomorphism classes of line bundles on X. We use the additive notation of divisors and multiplicative notation of line bundles interchangeably.

We use notation from [2], especially from Chapter 5. A basic reference for much of this material is [3, 1.5, 5.1, 5.2].

Let X be a surface and let $x \in X$. We will often work with the blow up $\pi : X_1 \to X$ of X at x. We denote the exceptional divisor of π by $E = \pi^{-1}(x)$. Then we have an isomorphism $Div(X_1) \cong Div(X) \oplus \mathbb{Z}$, where $Div(-)$ denotes the group of divisors modulo linear equivalence. The intersection pairing on $Div(X_1)$ is completely determined by the following (see [2, Proposition 5.3.2]).

(1) $\pi^\star(C) \cdot \pi^\star(D) = C \cdot D$ for $C, D \in Div(X)$;
(2) $\pi^\star(C) \cdot E = 0$ for $C \in Div(X)$;
(3) $E^2 = -1$.

The following is a nice argument for (3)[1].

Choose two curves C, D on X which meet transversally at x. Then the strict transforms of C, D are given by $\pi^\star(C) - E$ and $\pi^\star(D) - E$ respectively (see [2, Proposition 5.3.6]). Then using properties (1) and (2) above, $C \cdot D - 1 = C \cdot D + E^2$.

We recall the Riemann-Roch theorem for surfaces (see [2, Theorem 5.1.6]).

Theorem 1.1 (Riemann-Roch theorem for surfaces)**.** *Let X be a surface and let L be a line bundle on X. Suppose K denotes the canonical line bundle on X. Then* $h^0(X, L) - h^1(X, L) + h^2(X, L) = \frac{L\cdot(L-K)}{2} + 1 - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X)$.

We will also use the following theorem frequently (see [2, Theorem 5.1.10]).

Theorem 1.2 (Nakai criterion for ampleness)**.** *A line bundle L on X is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for all curves C on X.*

Theorem 1.3 (Kleiman's theorem)**.** *Let X be a surface. Let D be a divisor on X such that $D \cdot C \geq 0$ for every curve on X. Then $D^2 \geq 0$.*

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[1] We learned this from Pramathanath Sastry.

Proof. Fix an ample divisor H on X. For $t \in \mathbb{R}$, consider the quadratic polynomial $P(t) = (D+tH)^2 = D^2 + 2tD \cdot H + t^2H^2$. We are done if $P(0) \geq 0$. Suppose that $P(0) < 0$.

Since H is ample, there is a curve C in the linear system mH for some $m > 0$. By hypothesis on D, $D \cdot C = D \cdot mH \geq 0$, and hence $D \cdot H \geq 0$. Since the coefficient of t in $P(t)$ is non-negative and the coefficient of t^2 is positive, there is a real number $t_0 > 0$ such that $P(t_0) = 0$.

If $t > t_0$ is a rational number then $D + tH$ is ample. Indeed, this divisor meets all curves positively. Further $P(t) = (D+tH)^2 > 0$, so $D + tH$ is ample by Nakai criterion Theorem 1.2.

Now define $Q(t) = D \cdot (D + tH)$ and $R(t) = tH \cdot (D + tH)$, so that $P(t) = Q(t) + R(t)$ for every $t \in \mathbb{R}$. If $t > t_0$ is a rational number, $Q(t) \geq 0$ since $D + tH$ is ample. Thus $Q(t_0) \geq 0$ too, by continuity. On the other hand, $R(t_0) > 0$. We thus get $P(t_0) > 0$, which is a contradiction. □

Definition 1.4. A line bundle L is called *nef* if $L \cdot C \geq 0$ for every curve C on X.

2. Seshadri constants

Definition 2.1 (Seshadri constants)**.** Let X be a surface and let L be a nef line bundle on X. The *Seshadri constant of* L *at a point* $x \in X$, denoted $\epsilon(X, L, x)$, is defined as:

$$\epsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C}.$$

Remark 2.2. It suffices to consider only integral curves in Definiton 2.1. Indeed, it is easy to check that for two curves C and D passing through x, we have

$$\frac{L \cdot (C+D)}{\mathrm{mult}_x(C) + \mathrm{mult}_x(D)} \geq \min\left(\frac{L \cdot C}{\mathrm{mult}_x(C)}, \frac{L \cdot D}{\mathrm{mult}_x(D)}\right).$$

Lemma 2.3. *Let $\pi : X_1 \to X$ be the blow up of X at x with exceptional divisor E. Then*

$$\epsilon(X, L, x) = \sup\,\{\lambda \in \mathbb{R} \mid \pi^\star(L) - \lambda E \text{ is nef}\}.$$

Proof. Let ϵ_1 denote the supremum in the lemma and let $\epsilon = \epsilon(X, L, x)$.

Suppose that $\pi^\star(L) - \lambda E$ is nef for some λ and let C be a curve with multiplicity m at x. Then the strict transform of C is $\pi^\star(C) - mE$. Hence $(\pi^\star(L) - \lambda E) \cdot (\pi^\star(C) - mE) = L \cdot C - \lambda m \geq 0$. So $\lambda \leq \frac{L \cdot C}{m}$. This shows that $\epsilon_1 \leq \epsilon$. On the other hand, we claim that $\pi^\star(L) - \epsilon E$ is nef which proves that $\epsilon \leq \epsilon_1$ and the lemma is proved.

Let C_1 be an integral curve on X_1. If $C_1 = E$ then $(\pi^\star(L) - \epsilon E) \cdot E = \epsilon > 0$. Otherwise, C_1 is the strict transform of $C = \pi(C_1)$. So $C_1 = \pi^\star(C) - mE$, where m is the multiplicity of C at x. By definition, $\epsilon \leq \frac{L \cdot C}{m}$ which implies that $(\pi^\star(L) - \epsilon E) \cdot C_1 \geq 0$. □

Theorem 2.4 (Seshadri criterion for ampleness)**.** *A nef line bundle L on X is ample if and only if $\epsilon(X, L, x) > 0$ for every $x \in X$.*

Proof. Let D be a divisor on X corresponding to the line bundle L. If D is ample then a multiple mD is very ample for some positive integer m. Then mD is a hyperplane section for the embedding given by mD. If C is a curve on X passing through x, $mD \cdot C \geq \text{mult}_x(C)$ by properties of intersection numbers. Here we choose a hyperplane section which meets C properly. Hence $\epsilon = \epsilon(X, L, x) \geq \frac{1}{m}$.

Now assume that $\epsilon > 0$. We use Nakai criterion to show that D is ample. The hypothesis clearly implies that $D \cdot C > 0$ for any curve on X. We have to show that $D^2 > 0$. Choose an integer $a > 1/\epsilon$. We claim that $a\pi^\star(D) - E$ is nef on the blow up X_1 of X at x. Then by Theorem 1.3, $(a\pi^\star(D) - E)^2 = a^2D^2 - 1 \geq 0$, which proves that $D^2 > 0$.

In order to show that $a\pi^\star(D) - E$ is nef, we show that it meets all integral curves non-negatively. Since its intersection number with E is 1, we consider a curve $C_1 \neq E$ on X_1. Then C_1 is the strict transform of $C = \pi(C_1)$. So we may write $C_1 = \pi^\star(C) - mE$ where m is the multiplicity of C at x. So $(a\pi^\star(D) - E) \cdot C_1 = aD \cdot C - m \geq 0$, since $aD \cdot C \geq a\epsilon m \geq m$. ☐

Using one of the directions of the proof of Theorem 2.4, we obtain the following.

Corollary 2.5. *If L is very ample then $\epsilon(X, L, x) \geq 1$ for any $x \in X$.*

Remark 2.6. In fact, one can prove that $\epsilon(X, L, x) \geq 1$ for any $x \in X$ if L is merely ample and base point free. Indeed, let L be such a line bundle. Let $x \in X$ and let C be an irreducible and reduced curve on X passing through x. Then there exists a curve D in the linear system $|L|$ such that $x \in D$ and C is not a component of D. This is because the sections of L determine a finite morphism $f : X \to \mathbb{P}^n$, because L is ample. This means that $f(C)$ is not a point. So we can choose a hyperplane $H \subset \mathbb{P}^N$ such that $f(x) \in H$ and $f(C) \not\subset H$. Then $D := f^\star(H)$ has the required property. Then $L \cdot C = D \cdot C \geq \text{mult}_x(C)$.

Proposition 2.7 (An upper bound)**.** *Given X, L, x as in Definition 2.1, we have $\epsilon(X, L, x) \leq \sqrt{L^2}$.*

Proof. We use the equivalent formulation of $\epsilon = \epsilon(X, L, x)$ in Lemma 2.3. With the notation in Lemma 2.3, $\pi^\star(L) - \epsilon E$ is nef. By Theorem 1.3, $0 \leq (\pi^\star(L) - \epsilon E)^2 = L^2 - \epsilon^2$. ☐

Example 2.8. Let $p, q \in \mathbb{P}^2$ and let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at p, q. Let H be the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let E_1, E_2 be the exceptional curves. Let $L = 3H - E_1 - E_2$. Denote the strict transform of the line through p, q in $\mathbb{P}^2$. by l.

We claim that $\epsilon(X, L, x) = 1$ if $x \in E_1 \cup E_2 \cup l$ and $\epsilon(X, L, x) = 2$ otherwise.

It is not hard to see that L is ample and base point free (in fact L is very ample). Hence by Remark 2.6, $\epsilon(X, L, x) \geq 1$ for all $x \in X$.

Let $x \in E_1 \cup E_2 \cup l$. Then $\epsilon(X, L, x) = 1$, since $L \cdot E_1 = L \cdot E_2 = L \cdot l = 1$ and E_1, E_2, l are nonsingular.

Let $x \notin E_1 \cup E_2 \cup l$. Let l_1, l_2 be the strict transforms of the lines through $p, \pi(x)$ and $q, \pi(x)$. Since $L \cdot l_1 = 2$, we have $\epsilon(X, L, x) \leq 2$. We now show that $\epsilon(X, L, x) \geq 2$, to complete the argument.

Let C be any irreducible and reduced curve on X passing through x. Assume that C is different from E_1, E_2, l, l_1, l_2. Then C is the strict transform of $\pi(C)$ and we may write

$C = dH - m_1E_1 - m_2E_2$, where d is the degree of $\pi(C)$ and m_i are the multiplicities of $\pi(C)$ at p, q. Let m be the multiplicity of $\pi(C)$ at $\pi(x)$. Since $d > 1$ by assumption, we have $C \cdot l \geq 0$, $C \cdot l_1 \geq 0$ and $C \cdot l_2 \geq 0$. Hence $d \geq m_1 + m_2$, $d \geq m + m_1$ and $d \geq m + m_2$. This gives $3d \geq 2m + m_1 + m_2$, which in turn implies that $L \cdot C \geq 2m$. Hence $\epsilon(X, L, x) \geq 2$.

Definition 2.9. Let X be a surface.

(1) Let $x \in X$. The *Seshadri constant of* X *at* x is defined as $\epsilon(X, x) = \inf \epsilon(X, L, x)$, where the infimum is taken over all ample line bundles L on X.
(2) Let L be an ample line bundle on X. The *Seshadri constant of* L is defined as $\epsilon(X, L) = \inf \epsilon(X, L, x)$, where the infimum is taken over all points $x \in X$.
(3) The *Seshadri constant of* X is defined as $\epsilon(X) = \inf \epsilon(X, L)$, where the infimum is taken over all ample line bundles L on X. Equivalently, $\epsilon(X) = \inf \epsilon(X, x)$, where the infimum is taken over all points $x \in X$.

Question 2.10 (Some open questions about Seshadri constants)**.**

(1) Is there a surface X such that $\epsilon(X) = 0$?
(2) Is there a triple (X, L, x) such that $\epsilon(X, L, x)$ is irrational?

However the Seshadri constants can be arbitrarily small when we vary the surface.

Theorem 2.11 (Miranda's example)**.** *Given a real number* $\delta > 0$ *there exists a surface* X *and an ample line bundle* L *on* X *such that* $\epsilon(X, L, x) < \delta$ *for some* $x \in X$.

Proof. Choose $m \in \mathbb{N}$ so that $m > \frac{1}{\delta}$. For some $d > 2$, let $\Gamma \subset \mathbb{P}^2$ be an integral curve of degree d with a point $x \in \mathbb{P}^2$ of multiplicity m. Now choose another integral curve Γ' meeting Γ transversally such that the linear system spanned by Γ and Γ' consists of only integral curves. Such a Γ' exists by Lemma 2.12 below.

Let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\Gamma \cap \Gamma' = \{p_1, \ldots, p_{d^2}\}$. Denote by C and C' the strict transforms of Γ and Γ' respectively. Since p_i are all simple points on Γ and Γ', C, C' are isomorphic to Γ, Γ', respectively. In particular, C contains a point $x \in X$ of multiplicity m. Moreover the pencil $\langle C, C' \rangle$ spanned by C and C' consists of only integral curves.

We have a morphism $f : X \to \mathbb{P}^1$ determined by the linear system $\langle C, C' \rangle$. Note that f resolves the indeterminacies of the map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ determined by the pencil $\langle \Gamma, \Gamma' \rangle$. Let E be an exceptional divisor on X for the blow up π. For some $a \geq 2$, let $L = aC + E$. We claim that L is ample.

Since $C^2 = 0$, $C \cdot E = 1$ and $E^2 = -1$, we have $L^2 = 2a - 1 > 0$, $L \cdot C = 1 > 0$ and $L \cdot E = a - 1 > 0$. By Nakai, it suffices to show that $L \cdot D > 0$ for every curve D on X.

If D is a fibre of f then D is inside a curve in the pencil $\langle C, C' \rangle$. But since all curves in this linear system are integral, D itself is in it. So D is linearly equivalent to C and $L \cdot D = 1$. If D is not a fibre of f, let $p \in \mathbb{P}^1$. Since $f^{-1}(p)$ is a curve in the pencil $\langle C, C' \rangle$, it is linearly equivalent to C. As D is not in this linear system and it meets $f^{-1}(p)$ in a nonempty set $f^{-1}(p) \cap D$, we have $C \cdot D > 0$. Further it is clear that $D \cdot E \geq 0$. Indeed, if $D \cdot E < 0$ then E is a component of D. Since D is integral it follows that $D = E$. Thus $L \cdot D \geq a - 1 > 0$ and L is ample.

Finally, $\epsilon(X, L, x) \leq \frac{L \cdot C}{\mathrm{mult}_x(C)} = \frac{1}{m} < \delta$. □

Lemma 2.12. *Let $\Gamma \subset \mathbb{P}^2$ be an integral curve of degree $d > 2$. Then there exists an integral curve Γ' which intersects Γ transversally such that the linear system of plane curves spanned by Γ and Γ' contains only integral curves.*

Proof. The (projective) space of degree d plane curves has dimension $\binom{d+2}{2} - 1$. A degree d curve C is not integral precisely when there are polynomials f_1, f_2 such that $f = f_1 f_2$ where f is the form defining C. Thus the space of all non-integral degree d curves has dimension bounded above by $\binom{e+2}{2} + \binom{d-e+2}{2} - 2 = \frac{d^2+3d}{2} + e^2 - de$, for $e \in \{1, 2, \ldots, d-1\}$. This number is largest when $e = 1$ and in this case it is equal to $\frac{d^2+3d}{2} + 1 - d$. So the space of all non-integral degree d curves has codimension at least 1 in the space of all degree d curves. So a "general line" through Γ avoids this subspace. More precisely, there is a dense open set of integral curves Γ' such that the pencil spanned by Γ and Γ' consists only of integral curves.

Now by Bertini's theorem, there is a dense open subspace of degree d curves consisting of integral curves which meet Γ transversally. So we may choose Γ' satisfying the desired properties. □

In view of Proposition 2.7, we say that a Seshadri constant $\epsilon(X, L, x)$ is *sub-optimal* if $\epsilon(X, L, x) < \sqrt{L^2}$. We further say that a curve C is a *Seshadri curve* if $\epsilon(X, L, x) = \frac{L \cdot C}{\mathrm{mult}_x(C)}$. The following theorem says that there is always a Seshadri curve for a sub-optimal Seshadri constant. As a consequence, they are always rational.

Theorem 2.13. *Let X be a surface and let L be an ample line bundle on X. Let $x \in X$. Suppose that $\epsilon(X, L, x) < \sqrt{L^2}$. Then there is an irreducible curve C on X such that $\epsilon(X, L, x) = \frac{L \cdot C}{\mathrm{mult}_x(C)}$.*

Proof. The argument we give is due to Th. Bauer [1]. Let $\epsilon = \epsilon(X, L, x)$. Choose $\gamma \in \mathbb{Q}$ such that $\varepsilon < \gamma < \sqrt{L^2}$.

By definition of ϵ, there is a sequence $\{C_n\}_{n \geq 1}$ of irreducible curves such that $\frac{L \cdot C_n}{\mathrm{mult}_x(C_n)}$ converges to ϵ. Since $\epsilon < \gamma$, there exists some N such that $\frac{L \cdot C_n}{\mathrm{mult}_x(C_n)} < \gamma$ for $n \geq N$.

Next we make the following claim:

Claim: There exists an integer d such that there exists a curve $D \in |dL|$ containing x such that $\frac{L \cdot D}{\mathrm{mult}_x D} \leq \frac{L^2}{\gamma}$.

Proof: Let r be an integer such that $rL - K_X$ is ample. Note that since L is ample, such an integer exists. Now choose an integer $d > r$ such that $m := d\gamma$ is an integer.

By Riemann-Roch theorem and Serre vanishing, we have $h^0(dL) = \frac{dL \cdot (dL - K)}{2} + \chi(\mathcal{O}_X)$ (we may choose d large enough such that $h^1(dL) = h^2(dL) = 0$).

Re-writing the above equality, we have

$$\begin{aligned} h^0(dL) &= \frac{d(d-r)L^2}{2} + \frac{dL(rL-K)}{2} + \chi(\mathcal{O}_X) \\ &\geq \frac{d(d-r)L^2}{2} + \chi(\mathcal{O}_X). \end{aligned}$$

The linear system $|dL|$ contains a curve D which has multiplicity at least m at x provided $h^0(dL) \geq \frac{m^2+m}{2}$. Hence such a curve exists if $d(d-r)L^2 + 2\chi(\mathcal{O}_X) \geq m^2 + m \Leftrightarrow d(d-r)L^2 + 2\chi(\mathcal{O}_X) \geq d^2\gamma^2 + d\gamma \Leftrightarrow d^2(L^2 - \gamma^2) - d(rL^2 + \gamma) + 2\chi(\mathcal{O}_X) \geq 0$.

Since the last expression is a quadratic in d with a positive leading term, it is positive for large d. So we may choose such an integer $d > r$ such that $m = d\gamma$ is an integer. Thus there exists a curve $D \in |dL|$ such that $\mathrm{mult}_x(D) \geq m$.

Then $\frac{L\cdot D}{\mathrm{mult}_x(D)} \leq \frac{L\cdot D}{m} = \frac{dL^2}{d\gamma}$. So the claim holds.

The theorem now follows by the lemma below. The lemma implies that the collection of curves $\{C_n\}$ is finite. So the limit ϵ is achieved by some C_n. ☐

Lemma 2.14. *Given a real number $\gamma > 0$, suppose that there is a divisor $D \in |dL|$ for some $d > 0$, such that $\frac{L\cdot D}{\mathrm{mult}_x(D)} \leq \frac{L^2}{\gamma}$. Then every irreducible curve C satisfying $\frac{L\cdot C}{\mathrm{mult}_x(C)} < \gamma$ is a component of D.*

Proof. If C is not a component of D, then C and D meet properly. By Bezout's theorem, $D\cdot C = dL\cdot C \geq (\mathrm{mult}_x(D))(\mathrm{mult}_x(C)) > \frac{\gamma(L\cdot D)}{L^2}\frac{L\cdot C}{\gamma} = dL\cdot C$. This is a contradiction. ☐

2.1. **Multi-point Seshadri constants.** Let X be a smooth projective surface and let L be a nef line bundle on X. Let $r \geq 1$ be an integer. For $x_1, \ldots, x_r \in X$, the *multi-point Seshadri constant* of L at $x_1, \ldots, x_r$ is defined as follows.

$$\epsilon(X, L, x_1, \ldots, x_r) := \inf_{C\cap\{x_1,\ldots,x_r\}\neq\emptyset} \frac{L\cdot C}{\sum_i^r \mathrm{mult}_{x_i} C}.$$

As in case of the single point Seshadri constants, we note that the above infimum is the same as the infimum taken over irreducible, reduced curves C such that $C\cap\{x_1,\ldots,x_r\}\neq\emptyset$.

We have the following proposition which is an analogue of Proposition 2.7.

Proposition 2.15 (An upper bound)**.** *Given $X, L, x_1, \ldots, x_r$ as above, we have $\epsilon(X, L, x_1, \ldots, x_r) \leq \sqrt{\frac{L^2}{r}}$.*

Definition 2.16. Let X, L, r be as above. We set $\epsilon(X, L, r) := \max_{x_1,\ldots,x_r\in X} \epsilon(X, L, x_1, \ldots, x_r)$.

The value $\epsilon(X, L, r)$ is attained at a *very general* set of points $x_1, \ldots, x_r \in X$. For a reference, see [5]. Here *very general* means that $(x_1, \ldots, x_r)$ is outside a countable union of proper Zariski closed sets in $X^r = X\times X\times\ldots\times X$.

3. Examples

3.1. $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. We first consider the case of projective plane in detail.

If $r = 1$, it is easy to see that $\epsilon(X, L, x) = 1$ for any $x \in X$. Similarly, if $r = 2$ and x_1, x_2 are arbitrary points on X, then $\epsilon(X, L, x_1, x_2) = 1/2$.

Now let $x_1, x_2, x_3 \in X$. If the points are collinear, then the line through them is a Seshadri curve and $\epsilon(X, L, x_1, x_2, x_3) = 1/3$. On the other hand, if the points are not collinear, one has $\epsilon(X, L, x_1, x_2, x_3) = 1/2$. Indeed, let l be the line through x_1 and x_2. By hypothesis, $x_3 \notin l$. Hence the Seshadri quotient for l is $1/2$. Let $C \neq l$ be any other irreducible curve of degree d passing through at least one of the points x_1, x_2 or x_3. For $i = 1, 2, 3$, let m_i denote the multiplicity of C at x_i. It is clear that $d \geq m_i$ for every i. Applying Bezout to the irreducible curves C and l, we have $d \geq m_1 + m_2$. Hence we obtain $2d \geq m_1 + m_2 + m_3$. Hence $\epsilon(X, L, x_1, x_2, x_3) = 1/2$.

Thus we have $\epsilon(X, L, 3) = 1/2$, and this is achieved for any three non-collinear points.

Similarly, it is not hard to prove that $\epsilon(X, L, 4) = 1/2$, $\epsilon(X, L, 5) = \epsilon(X, L, 6) = 2/5$, $\epsilon(X, L, 7) = 3/8$, $\epsilon(X, L, 8) = 6/17$ and $\epsilon(X, L, 9) = 3$.

For $r \geq 10$, the value of $\epsilon(X, L, r)$ is not known except when r is a square. If $r = s^2$ and $s \geq 3$, Nagata [4] showed that $\epsilon(X, L, r) = 1/s$. In the same paper, Nagata made the following famous conjecture.

Conjecture 3.1 (Nagata Conjecture). *$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r) = \frac{1}{\sqrt{r}}$, when $r \geq 10$.*

Note that an affirmative solution to Nagata Conjecture for even one non-square value of $r \geq 10$ also solves Question 2.10 (2).

Nagata Conjecture 3.1 is a statement about linear systems of plane curves in the sense we explain now.

Let $x_1, \ldots, x_r$ be very general points of $\mathbb{P}^2$. Let $d, m_1, \ldots, m_r$ be non-negative integers. We are interested in the question:

Is there a curve $C \in \mathbb{P}^2$ of degree d passing through x_i with multiplicity at least m_i for each $1 \leq i \leq r$?

Let $\mathcal{L}(d, m_1, \ldots, m_r)$ denote the linear system of plane curves of degree d passing through x_i with multiplicity at least m_i for each $1 \leq i \leq r$. So our question can be rephrased as:

Is $\mathcal{L}(d, m_1, \ldots, m_r)$ non-empty?

The linear system of degree d plane curves has dimension $\frac{d(d+3)}{2}$. This can be seen by observing that a degree d plane curve is given by a homogeneous form of degree d in three variables. This is also a special case of Riemann-Roch Theorem 1.1.

On the other hand, a point of multiplicity m imposes $\binom{m+1}{2}$ conditions on the linear systems of curves of degree d. It follows that $\mathcal{L}(d, m_1, \ldots, m_r)$ is non-empty provided $\frac{d(d+3)}{2} \geq \sum_{i=1}^{r} \binom{m_i+1}{2}$. Let us refer to this as the Riemann-Roch inequality.

It is easy to prove that if $d, m_1, \ldots, m_r$ satisfy the Riemann-Roch inequality, then the Seshadri quotient $\frac{d}{\sum_i m_i}$ is at least $\frac{1}{\sqrt{r}}$, as predicted by the Nagata Conjecture. However this inequality is not necessary for existence of such curves, as the example below shows.

Consider $r = 11$, $d = 10$, $m_1 = 8$, and $m_2 = \ldots = m_{11} = 2$. Then $\frac{d(d+3)}{2} = 65$ while $\sum_{i=1}^{11} \binom{m_i+1}{2} = 66$. However there is a curve in the linear system $\mathcal{L}(10, 8, 2, \ldots, 2)$! Indeed, we first observe that $\mathcal{L}(5, 4, 1, \ldots, 1)$ is non-empty (again $r = 11$). This follows from Riemann-Roch inequality. Let $C \in \mathcal{L}(5, 4, 1, \ldots, 1)$. Then $2C \in \mathcal{L}(10, 8, 2, \ldots, 2)$. Note that the Seshadri quotient for $2C$ is $\frac{10}{28} = 0.3571$, while the Nagata Conjecture predicts that $\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), 11) = \frac{1}{\sqrt{11}} = 0.3015$. The curve C is an example of *exceptional curve*: its proper transform on the blow up of $\mathbb{P}^2$ at $x_1, \ldots, x_{11}$ is a smooth rational curve of self-intersection -1.

So, in order to prove the Nagata Conjecture, one needs to understand effective curves given by $d, m_1, \ldots, m_r$ that do *not* satisfy the Riemann-Roch inequality.

There is a precise conjecture, called the *SHGH Conjecture*, which completely characterizes tuples $(d, m_1, \ldots, m_r)$ which represent effective plane curves. It says, in part, that any effective linear system of plane curves given by a tuple $(d, m_1, \ldots, m_r)$ not satisfying the Riemann-Roch inequality contains a non-reduced multiple of an exceptional curve. The SHGH Conjectures implies the Nagata Conjecture.

References

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