Linear Algebraic Groups, Midterm exam

All problems are worth 10 points.

(1) Let G be a connected algebraic group and let N be a finite normal subgroup of G. Show that N is contained in the center of G.

(2) Show that any morphism of algebraic varieties $\mathbb{G}_a \to \mathbb{G}_m$ is constant.

(3) Let $X = \mathbb{A}^1 \setminus \{0\}$.

(a) Find $Hom_{var}(X, X)$, the group of morphisms of varieties from X to itself.

(b) Find $Hom_{ag}(X, X)$, the group of morphisms of algebraic groups from X to itself.

Hint: Consider the ring homomorphisms from $K[X]$ to itself.

(4) Let $G = SL(2, K)$ and let $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Using the following steps compute the centralizer $C_G(x)$ of x and show that it is not connected.

First, find $C_G(x)$ and show that H is a closed subgroup of $C_G(x)$ of index 2, where $H = \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) | a \in K \right\}$. Then use the fact that closed subgroups of finite index contain the identity component.

(5) Let G be an algebraic group acting morphically on an affine variety X. Show that there exists a finite dimensional vector space V such that

(a) G acts morphically on V; i.e., there is a morphism of varieties $G \times V \to V$, and

(b) there is a G-equivariant morphism $\phi : X \to V$ such that ϕ is an isomorphism onto a closed G-stable subvariety Y of V.

Hint: Use the fact that any finite dimensional K-subspace of $K[X]$ is contained in a G-stable finite dimensional subspace of $K[X]$.