Linear Algebraic Groups, Homework 1

(1) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain (hint: it is isomorphic to $\mathbb{C} \times \mathbb{C}$).
(2) Let $\phi : X \to Y$ be a morphism of affine varieties. Suppose that $\phi(X)$ is dense in Y. Show that the morphism of rings $\phi^\star : K[Y] \to K[X]$ is injective.
(3) Show that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic.
(4) Prove that the only automorphisms (as algebraic groups) of $\mathbb{G}_m$ are identity and inverse maps and Aut $\mathbb{G}_a \cong K^\star$.
(5) Prove that the G^0 is stable under all automorphisms of G.
(6) Let A be a finite dimensional K-algebra (dimension of A as a K-vector space is finite). Consider the group $Aut(A)$ of algebra automorphisms of A. Show that $Aut(A)$ is a closed subgroup of $GL(A)$. Hence $Aut(A)$ is an algebraic group.
(7) Is $GL(n, \mathbb{R})$ a closed subgroup of $GL(n, \mathbb{C})$?
(8) Let $N \subset GL(n, K)$ be the group of **monomial** matrices (matrices having exactly one nonzero entry in each row and column). Prove that N is a closed subgroup of $GL(n, K)$, with $N^0 = D(n, K)$ and $[N : N^0] = n!$.
(9) Let X be an irreducible affine variety over $\mathbb{C}$ of positive dimension. Show that X **can not** be written as a union of countably many proper closed subsets of X. (First use induction on n to prove this for $X = \mathbb{A}^n$ and then use Noether's normalization lemma for arbitrary X).
(10) Let G be a closed subgroup of $GL(n, K)$ (i.e, G is a *linear algebraic group*). For any $m \geq 1$, let G_m be the subset of G consisting of elements of order m. Show that G_m closed in G for all m (first show this for $GL(n, K)$; think eigenvalues and diagonalizaion).
(11) Let G be a complex linear algebraic group such that every element of G has finite order. Show that G is finite (use previous two problems).