Topology, Problem Set 9

Definition 1: A space is **locally metrizable** if each point x of X has a neighborhood that is metrizable (in the subspace topology).
Definition 2: Let Y be a normal space. Y is said to be an **absolute retract (AR)** if whenever there is an imbedding $h : Y \to Z$ into a normal space Z such that $h(Y)$ is closed in Z, then $h(Y)$ is a retract of Z.
Definition 3: A space Y is said to have the **universal extension property** if for each triple consisting of a normal space X, a closed subset A of X and a continuous function $f : A \to Y$, there is a an extension of f to a continuous map of X into Y.
Definition 4: A space X is $\sigma-$**compact** if it is a union of countably many compact sets.

(1) Give an example of a space that is Hausdorff with a countable space, but not metrizable.
(2) Give an example of a countable space is that not first countable.
(3) Let X be a compact Hausdorff space. If X is second countable, is it true that X is metrizable? What about the converse?
(4) Let X be a locally compact Hausdorff space.
  (a) Is it true that if X has a countable basis, then X is metrizable? What about the converse?
  (b) Let Y be the one-point compactification of X. Is it true that if X has a countable basis, then Y is metrizable? What about the converse?
(5) Show that a compact Hausdorff is metrizable if it is locally metrizable. (Hint: Show that X is a finite union of open subspaces, each of which has a countable basis.)
(6) show that $\mathbb{R}$ has the universal extension property.
(7) Suppose that X has the universal extension property and $Y \, subset X$ is a retract of X. Show that Y has the universal extension property.
(8) Let X be a normal space and let Y be a closed subset of X. Suppose that Y has the universal extension property, show that Y is a retract of X.
(9) Show that if Y is compact then the following are equivalent:
  (a) Y is homeomorphic to a retract of $[0, 1]^J$ for some J.
  (b) Y is homeomorphic to a retract of $\mathbb{R}^J$ for some J.
  (c) Y has the universal extension property.
  (d) Y is an absolute retract.
(10) Show that a separable locally compact metric space is $\sigma-$compact.
(11) Show that a $\sigma-$compact metric space is separable.