Topology, Problem Set 8

Definition 1: A space X is said to be **completely normal** if every subspace of X is normal.
Definition 2: Two subsets A, B of a space X are called **separated** if $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.
Definition 3: Let X be a space and let $Y \subset X$ be a subspace. We say that Y is a **retract** of X if there is a continuous map $r : X \to Y$ such that $r(y) = y$ for $y \in Y$.

(1) Show that if X is regular then every pair of points of X have neighbourhoods whose closures are disjoint.
(2) Show that every order topology is regular.
(3) Show that a closed subset of a normal space is normal.
(4) Show that every regular Lindelof space is normal.
(5) Is $\mathbb{R}^\omega$ normal in the product topology?
(6) Show that X is completely normal if and only if for every pair of separated sets A and B, there exist disjoint open sets containing them.
(7) Which of the following spaces are completely normal?
  (a) A subspace of a completely normal space.
  (b) The product of two completely normal spaces.
  (c) A well-ordered set in the order topology.
  (d) A metric space.
  (e) A compact Hausdorff space.
  (f) A regular space with a countable basis.
  (g) The space $\mathbb{R}_l$.
(8) Let $X = I^I$, where $I = [0, 1]$ (with product topology). We note that X is compact using Tychonoff theorem. It is also Hausdorff. So X is normal. However, show that X is not metrizable. This shows that normality does not characterize metrizability.
(9) Show that every metrizable space is first countable. Give an example of a metric space that is **not** second countable.
(10) Show that the Tietze extension theorem implies the Urysohn lemma.
(11) Show that a connected regular space having more than one point is uncountable.
(12) Show that a connected normal space having more than one point is uncountable.
(13) Give an example of a connected, Hausdorff, countable space.
(14) Let X be a regular space with a countable basis. Let $U \subset X$ be open.
  (a) Show that U equals a countable union of closed sets of X.
  (b) Show that there is continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.
(15) (a) Show that X is Hausdorff and Y is a retract of X, then Y is closed in X.
  (b) Let A be a two-point set in $\mathbb{R}^2$. Show that A is **not** a retract of $\mathbb{R}^2$.
  (c) Let S^1 be the unit circle in $\mathbb{R}^2$. Show that S^1 is a retract of $\mathbb{R}^2 - \{0\}$. What is your guess regarding S^1 being a retract of $\mathbb{R}^2$?