Topology, Problem Set 7

Definition: Let (X, d) be a metric space. A continuous map $f : X \to X$ is called a **contraction** if there exists a number $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. We say that f is an **isometry** if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

(1) Show that the space $[0, 1]^\omega$ is not limit point compact in the box topology.
(2) Consider $\mathbb{N}$ with discrete topology and $Y = \{a, b\}$ with indiscrete topology. Let $X = Y \times \mathbb{N}$ have the product topology. Show that X is not compact, but limit point compact.
(3) Show that $[0, 1]$ is not limit point compact as a subspace of $\mathbb{R}_l$.
(4) Show that a countably compact space is limit point compact.
(5) If f is a contraction on a compact metric space X, show that there is a unique point $x \in X$ such that $f(x) = x$.
(6) If f is an isometry on a compact metric space, show that f is surjective.
(7) Suppose that X is second countable. Show that every collection of disjoint open sets in X is countable.
(8) Let X be a second countable Hausdorff space. Show that all four varieties of compactness (compactness, limit point compactness, sequential compactness and countable compactness) are equivalent.
(9) If X is Lindelof and Y is compact, then show that $X \times Y$ is Lindelof.
(10) Show that a metric space is first countable.
(11) Show that a compact metric space is separable.
(12) Let X be a metric space. Show that X is separable if and only if it is second countable.
(13) Let X be the set of all integers greater than 1. For $n \geq 2$, let U_n be the set of all divisors of n in X. Show that U_n is a basis for a topology on X. Show that under the topology generated by this basis
   (a) X is T_0, but not Hausdorff;
   (b) X is not countably compact;
   (c) X is second countable;
   (d) X is connected and locally connected.