Topology, Problem Set 6

Definition: A collection $\mathcal{C}$ of closed subsets of a space X is said to satisfy the **finite intersection property** if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \ldots \cap C_n$ is nonempty.

(1) Is $\mathbb{R}$ compact in the finite complement topology?

(2) Define a topology on $\mathbb{R}$ by declaring that a set is open if it is empty or its complement is countable. Is $\mathbb{R}$ compact in this topology? What about the subspace $[0,1]$?

(3) Show that every compact subset of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded set is compact.

(4) Let A and B be compact sets in a Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B respectively.

(5) Suppose that $f : X \to Y$ is continuous with X compact and Y Hausdorff. Show that f is a closed map.

(6) Suppose that Y is compact. Then show that the projection map $\pi_1 : X \times Y \to X$ is closed.

(7) Show that X is compact if and only if for every collection $\mathcal{C}$ of closed subsets of X satisfying the finite intersection property, the intersection $\cap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

(8) Let X be an ordered set in which every closed interval is compact. Show that X has the least upper bound property.

(9) Let X be compact Hausdorff space. Show that if $\{A_n\}$ is a countable collection of closed sets in X, each of which has empty interior in X, then there is a point of X which is not in any A_n.

(10) Let $A_0 = [0,1]$ in $\mathbb{R}$. Let A_1 be obtained from A_0 by deleting the "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general. define A_n by:

$$A_n = A_{n-1} - \bigcup_{k=0}^{n-1} \left(\frac{1+3^k}{3^n}, \frac{2+3^k}{3^n}\right).$$

Now let $C = \cap_n A_n$. C is called the Cantor set. Show that.

(a) C is totally disconnected;

(b) C is compact;

(c) each set A_n is a union of finitely many disjoint closed intervals of length $\frac{1}{3^n}$ and the end points of these intervals lie in C.

(d) every point of C is a limit point of C;

(e) C is uncountable;

(11) Show that the space $\prod_{n=1}^{\infty} [0,1]$ is not compact in box topology.

(12) A **countably compact** space is a space in which every countable open cover has a finite subcover. Prove that a second countable space is countably compact if and only if it is compact.

(13) Let $\{A_n\}$ be a *nested* collection of subsets of a space X; that is $A_1 \supset A_2 \supset A_3 \supset \ldots$. Assume that each A_n is nonempty and compact. Show that $\cap_n A_n$ is nonempty.

(14) Let $f : X \to Y$ be a bijective continuous map with X compact and Y Hausdorff. Show that f is a homeomorphism.
(15) Suppose that a compact metric space X is a union of two of its open sets U and V. Prove that there exists a real number δ such that every subset of X of diametre less than δ is contained in U or V.
(16) Show that in a Hausdorff space arbitrary intersection of compact sets is compact.
(17) Show that a finite union of compact sets in a space X is compact.