**Topology, Problem Set 5**

(1) What are the components and path components of $\mathbb{R}_l$?
(2) What are the components and path components of $\mathbb{R}^\omega$, in the product topology?
(3) If $f : X \to Y$ is continuous and $X$ is locally connected, then is $f(X)$ necessarily locally connected? What if $f$ is also open? (recall that an open map is a map $f$ such that $f(U)$ is open for all open $U$. Similarly a closed map is a map that sends closed sets to closed sets.)
(4) Show that $[0,1] \times [0,1]$ in the dictionary topology is locally connected but not locally path connected. What are the path components of this space?
(5) Show that every connected open subset of a locally path connected space is path connected.
(6) Let $X$ denote the set of rational points of $[0,1] \times 0$ in $\mathbb{R}^2$. Let $T$ denote the union of all line segments joining $p = 0 \times 1$ to points of $X$.
(a) Show that $T$ is path connected, but is locally connected only at the point $p$.
(b) Find a subset of $\mathbb{R}^2$ that is path connected but is locally connected at none of its points.
(7) A space $X$ is said to be **connected im kleinen at** $x$ if for every neighbourhood $U$ of $x$, there is a connected subset $A$ of $U$ that contains a neighbourhood of $x$. Show that if $X$ is connected im kleinen at each of its points, then $X$ is locally connected.
(8) Suppose that $f : X \to Y$ is a continuous, open and closed map. Suppose that $Y$ is connected. Show that $f$ is surjective.
(9) Let $X$ be the subset of $[0,1] \times [0,1]$ consisting of points $a \times b$ where $a, b$ are both irrational numbers. Consider $X$ with subspace topology (in $\mathbb{R}^2$). Is $X$ connected? path connected? locally connected?
(10) Show that any open connected subset of $\mathbb{R}^n$ is path connected.
(11) Show that a circle in $\mathbb{R}^2$ is not homeomorphic to an open interval in $\mathbb{R}$. Similarly, show that a circle can not be homeomorphic to a closed interval.