Topology, Problem Set 4

Definition: Let $f : X \to Y$ be an injective continuous map with image Z. f is called an **imbedding** if the map $f : X \to Z$ is a homeomorphism.

(1) Show that no two of the spaces (0,1), (0,1], and [0,1] are homeomorphic.
(2) Suppose that $f : X \to Y$ and $g : Y \to X$ are imbeddings. Are X and Y homeomorphic?
(3) Let $f : X \to X$ be a continuous map. If $X = [0, 1]$, show that there is a point $x \in X$ such that $f(x) = x$. x is called a *fixed point* of f. Is there a fixed point when $X = [0, 1)$ or $X = (0, 1)$?
(4) Let X be an ordered set with order topology. Show that if X is connected, then X is a linear continuum.
(5) (a) Is a product of path-connected spaces path-connected?
(b) If $A \subset X$ and A is path-connected, is $\bar{A}$ path-connected?
(c) If $f : X \to Y$ is continuous and X path-connected, then is $f(X)$ path-connected?
(d) Is the union of a collection of path-connected subsets of X path-connected, if the intersection of the collection is nonempty?
(6) Show that $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.
(7) Let $A \subset \mathbb{R}^2$ be countable. Show that $\mathbb{R}^2 - A$ is path-connected.
(8) Let $U \subset \mathbb{R}^2$ be open and connected. Is U path-connected?
(9) Let X be a topological space. A connected subspace Y of X is *maximal* if it is not properly contained in a larger connected subspace. If Y is maximal, then it is called a **connected component** of X. Show the following:

(a) A connected subspace of X that is both closed and open is a connected component.
(b) A connected component of X is closed.
(c) Let $X = \mathbb{Q}$. Find the connected components of X. Use this to show that connected components need not be open.