Topology, Problem Set 3

(1) Show that if $\prod_\alpha X_\alpha$ is connected and nonempty, then each X_α is connected.

(2) Show that $\mathbb{R}^\omega$ is not connected in box topology by showing that the set A of all bounded sequences is both open and closed. What happens if $\mathbb{R}^\omega$ has uniform topology?

(3) A space is **totally disconnected** if its only connected subsets are one-point sets. Show that a finite Hausdorff space is totally disconnected.

(4) Is it true that if X has discrete topology, then X is totally disconnected? Does the converse hold?

(5) Show that an infinite set X is connected in finite complement topology. In fact, show that any two nonempty open sets in X have nonempty intersection.

(6) Is $\mathbb{R}_l$ connected?

(7) Let $\{A_n\}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\cup A_n$ is connected.

(8) Show that X is connected if and only if for every two points in X, there is some connected subspace of X which contains both.

(9) Let X be connected. Suppose that there is a nonconstant continuous function $f : X \to \mathbb{R}$. Show that X is uncountable.

(10) Show that a Banach space is connected (refer to PS 2 for a definition) by writing it as a union of connected subspaces with mutually nonempty intersection.

(11) Suppose that X has a countable basis (we say in this case that X is **second countable**). Show that X is separable.

(12) Show that the following subspaces are closed:
  (a) $A = \{(x, y) | xy = 1\}$ in $\mathbb{R}^2$;
  (b) $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ in $\mathbb{R}^2$;
  (c) $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$;
  (d) $B^2 = \{(x, y) | x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$.

  S^1 is called a **circle**, S^2 is called a **sphere** and B^2 is called a **ball**.

(13) Determine if the following pairs of spaces are homeomorphic.
  (a) $\mathbb{R}$ and $\mathbb{R}^2$;
  (b) $S^1 - \{(0, 1)\}$ and $\mathbb{R}$;
  (c) $S^2 - \{(0, 0, 1)\}$ and $\mathbb{R}^2$;
  (d) $\mathbb{R}$ and $\mathbb{R} - \{10\}$;
  (e) $\mathbb{R}_l$ and $\mathbb{R}$.

(14) On Wednesday, 18th January, in class, we argued that a product of connected spaces is connected and our argument seemed to work for both product and box topologies. But this is not true :(– see problem (2). Find out what is wrong in our "proof".