Topology, Problem Set 2

Definition 1: Let X be a vector space over $\mathbb{R}$. A **norm** on X is a function $|| \ || : X \to \mathbb{R}$ such that:

(a) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0 \Leftrightarrow x = 0$;
(b) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$;
(c) $||rx|| = |r|||x||$ for $r \in \mathbb{R}$ and $x \in X$.

A vector space with a norm is called a **normed vector space**.

(1) Show that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity is equivalent to the open set definition.
(2) Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic to $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic to $[0, 1]$
(3) Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.
(4) Let $A \subset X$. Let $f : A \to Y$ be a continuous map. Let Y be Hausdorff. Show that if f may be extended to a continuous map $g : \bar{A} \to Y$, then g is uniquely determined by f.
(5) Let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \cup_\alpha A_\alpha$. Suppose that $f : X \to Y$ is a map such that $f|A_\alpha$ is continuous for all α.
  (a) Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.
  (b) Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.
  (c) An indexed family of sets $\{A_\alpha\}$ is called **locally finite** if each point x of X has a neighbourhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.
(6) Let $f : X \to \mathbb{R}$ be continuous. Let $Y = \{x \in X | f(x) \neq 0\}$. Suppose that Y is nonempty (we say that f is not *identically zero*). Prove that the function $1/f : Y \to \mathbb{R}$ defined by $(1/f)(x) = 1/f(x)$ is continuous.
(7) Let $\{X_\alpha\}$ be a collection of topological spaces and $X = \prod_\alpha X_\alpha$. Show that the product topology is the coarsest (smallest) topology on X relative to which each projection map $\pi_\alpha : X \to X_\alpha$ is continuous.
(8) Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are *eventually* zero: that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ (in both box and product topologies)?
(9) A topological space X is called **separable** if it has a countable dense subset. Show that $\mathbb{R}^n$ and $\mathbb{C}^n$ are separable. What about $\mathbb{R}^J$ for an arbitrary index set J (under box and product topologies)?
(10) Let X be a normed vector space over $\mathbb{R}$. Define a function $d : X \times X \to \mathbb{R}$ by $d(x, y) = ||x - y||$. Show that d is a metric on X. Thus every normed vector space has a natural structure of a metric space.

Definition 2: A metric space X is **complete** if every Cauchy sequence in X converges.
Definition 3: A normed vector space is called a **Banach space** if it is complete as a metric space.

(11) Is $\mathbb{R}$ a Banach space?

(12) Let S be a nonempty set. A function $f : S \to \mathbb{R}$ is **bounded** if there exists a $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in S$. Equivalently, $\sup_{x \in S}|f(x)| < \infty$.

Let X be the set of all bounded real functions on S. Then X has a natural structure of a real vector space. Define a function $|| \; || : X \to \mathbb{R}$ by setting $||f|| = \sup_{x \in S}|f(x)|$. Show that this defines a norm on X. Then show that X is a Banach space.