Topology, Problem Set 11

Definition 1: A space X is said to be **contractible** if the identity map $i_X : X \to X$ is homotopic to some constant map $X \to X$.
Definition 2: A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all the line segments joining a_0 to other points of A lie in A.

(1) Suppose that X is a convex set in $\mathbb{R}^n$; that is: for all $x, y \in X$, the line segment joining them lies in X. Show that any two paths in X having the same end points are path homotopic.
(2) Given spaces X, Y, let $[X, Y]$ denote the set of homotopy classes of continuous maps of X into Y.
  (a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.
  (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.
(3) Show that I and $\mathbb{R}$ are contractible.
(4) Show that a contractible space is path connected.
(5) Show that, if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
(6) Show that, if X is contractible and Y is path connected, then $[X, Y]$ has a single element.
(7) Find a star convex set that is not convex.
(8) Show that if A is star convex, then it is simply connected.
(9) Show that if A is star convex, any two paths in A having same initial and final points are path homotopic.
(10) Let x_0, x_1 be two points in a path connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.
(11) Let $A \subset X$ and let $r : X \to A$ be a retraction. Given $a_0 \in A$, show that $r_\star : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.
(12) Let A be a subset of $\mathbb{R}^n$ and let $h : (A, a_0) \to (Y, y_0)$ be a continuous map. Show that if h is extendable to a continuous map of $\mathbb{R}^n$ into Y, then $h_\star$ is the zero homomorphism.
(13) Let X be a space. Show that the following are equivalent.
  (a) Every continuous map $S^1 \to X$ is homotopic to a constant map.
  (b) Every continuous map $S^1 \to X$ extends to a continuous map $B^2 \to X$.
  (c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.