Topology, Problem Set 1

Definition 1: Let X be a topological space and let $A \subset X$. Define the *interior* of A, written A°, as the union of all open sets contained in A. Define the *closure* of A, written $\bar{A}$, as the intersection of all closed sets containing A. Define the *boundary* of A, written Bd A, as $\bar{A} \cap \overline{(X - A)}$.

Definition 2: Let X be a topological space and let $x \in X$ and $A \subset X$. We say that x is a *limit point* of A if every neighbourhood of x intersects A in some point other than x itself.

Definition 3: A topological space X is said to be T_1 if given any two distinct points $x, y \in X$, each has a neighbourhood not containing the other.

(1) Let X be a set. Consider the collection of subsets 0f X consisting of the empty set and all sets whose complements are countable. Is this a topology on X?

(2) Show that if a topological space is metrizable, then it is metrizable in an infinite number of different ways (by different metrics).

(3) Let X be a topological space with a specified basis. Let $x \in X$ and $A \subset X$. Show that $x\ in \bar{A}$ if and only if every basic open set B containing x intersects A.

(4) Let X be a topological space and let $A \subset X$. Let A' be the set of all limit points of A. Show that $\bar{A} = A \cup A'$.

(5) Consider $\mathbb{R}$ with the standard topology. Let $Y = (-1, 1]$ have the 'subspace topology. What is the closure of $(-1, 1/2)$ in $\mathbb{R}$ and in Y?

(6) Show that a subspace of a Hausdorff space is Hausdorff. Show that a product of two Hausdorff spaces is Hausdorff.

(7) Show that a topological space X is Hausdorff if and only if the *diagonal* $\Delta = \{(x, x)|x \in X\}$ is closed in $X \times X$ (under product topology).

(8) Let X be a topological space and let $A \subset X$. Show that A° and Bd A are disjoint, and $\bar{A} = A^\circ \cup$ Bd A.

(9) Show that the finite complement topology on $\mathbb{R}$ is not Hausdorff.

(10) Show that a finite subset of a Hausdorff space is closed.

(11) A subset $A \subset X$ is called *dense* if its closure is X. Show that A is dense if and only if the only closed superset of A is X if and only if every nonempty open set has nonempty intersection with A.

(12) A subset $A \subset X$ is *nowhere dense* if its closure has empty interior. Show that a closed subset $V \subset X$ is nowhere dense if and only if its complement is dense.

(13) Give an example of a set in a topological space which
  (a) is both open and closed;
  (b) is neither open nor closed;
  (c) contains a point which is not a limit point of the set;
  (d) contains no point which is not a limit point of the set.

(14) Give an example of a T_1 space that is not Hausdorff.

(15) A sequence $\{x_n\}$ of points in a topological space X is said to be *convergent* if there exists point $x \in X$ such that for each neighbourhood U of x, there exists a positive integer N (depending on U) such that $x_n \in U$ for $n \geq N$. The point x is called a *limit point* of the sequence $\{x_n\}$.

Now let X be Hausdorff. Show that every convergent sequence in X has a unique limit point. Is this true for non-Hausdorff spaces?