EFFECTIVE CONE OF A GRASSMANN BUNDLE OVER A CURVE DEFINED OVER $\overline{\mathbb{F}}_p$

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Abstract. Let X be an irreducible smooth projective curve defined over $\overline{\mathbb{F}}_p$ and E a vector bundle on X of rank at least two. For any $1 \leq r < \text{rank}(E)$, let $\text{Gr}_r(E)$ be the Grassmann bundle over X parametrizing all the r dimensional quotients of the fibers of E. We prove that the effective cone in $\text{NS}(\text{Gr}_r(E)) \otimes_{\mathbb{Z}} \mathbb{R}$ coincides with the pseudo-effective cone in $\text{NS}(\text{Gr}_r(E)) \otimes_{\mathbb{Z}} \mathbb{R}$. When $r = 1$ or $\text{rank}(E) - 1$, this was proved in [Mo].

1. introduction

Let X be an irreducible smooth projective curve defined over an algebraically closed field k, and let E be a vector bundle over X of rank N, with $N > 1$. Fix an integer $1 \leq r \leq N - 1$, and denote by $\text{Gr}_r(E)$ the Grassmann bundle over X parametrizing all the r dimensional quotients of the fibers of E. The Néron–Severi group $\text{NS}(\text{Gr}_r(E))$ is the group of divisors on $\text{Gr}_r(E)$ modulo algebraic equivalence, so $\text{NS}(\text{Gr}_r(E))$ coincides with the group of connected components of the Picard group $\text{Pic}(\text{Gr}_r(E))$. The pseudo-effective cone of $\text{NS}(\text{Gr}_r(E))_{\mathbb{R}} := \text{NS}(\text{Gr}_r(E)) \otimes_{\mathbb{Z}} \mathbb{R}$ is the closure of the effective cone of $\text{NS}(\text{Gr}_r(E))_{\mathbb{R}}$. In [BHP] the pseudo-effective cone of $\text{NS}(\text{Gr}_r(E))_{\mathbb{R}}$ was computed.

Let p be a prime number, and let $\mathbb{F}_p$ be the field of order p. When k is the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, Moriwaki proved that the pseudo-effective cone of $\text{NS}(\text{Gr}_1(E))_{\mathbb{R}}$ coincides with the effective cone of $\text{NS}(\text{Gr}_1(E))_{\mathbb{R}}$ [Mo, p. 802, Theorem 0.4]. Replacing E by its dual E^* it is deduced from this that the pseudo-effective cone of $\text{NS}(\text{Gr}_{N-1}(E))_{\mathbb{R}}$ coincides with the effective cone of $\text{NS}(\text{Gr}_{N-1}(E))_{\mathbb{R}}$.

Our aim here is to prove the following (see Theorem 3.1):

Theorem: *Let $k = \overline{\mathbb{F}}_p$ for a prime p. The pseudo-effective cone of $\text{NS}(\text{Gr}_r(E))_{\mathbb{R}}$ coincides with the effective cone of $\text{NS}(\text{Gr}_r(E))_{\mathbb{R}}$ for all $1 \leq r \leq N - 1$.*

2. Pseudo-effective cone of a Grassmann bundle

Let k be an algebraically closed field. Let X be an irreducible smooth projective curve defined over k. Take any vector bundle E over X such that

$$N := \text{rank}(E) \geq 2. \tag{2.1}$$

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Fix an integer $1 \leq r \leq N-1$. Let

$$\phi \,:\, \mathrm{Gr}_r(E) \longrightarrow X \tag{2.2}$$

be the Grassmann bundle parametrizing the r dimensional quotients of the fibers of E. Let $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ be the tautological relatively ample line bundle over $\mathrm{Gr}_r(E)$. The fiber of $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ over the point of $\mathrm{Gr}_r(E)$ representing a quotient $E_x \longrightarrow Q$, where $x \in X$, is $\bigwedge^r Q$. Fix a line bundle L on X of degree one. The Néron–Severi group $\mathrm{NS}(\mathrm{Gr}_r(E))$ of $\mathrm{Gr}_r(E)$, which is the group of connected components of the Picard group $\mathrm{Pic}(\mathrm{Gr}_r(E))$, is the free abelian group generated by the classes of $\mathcal{O}_{\mathrm{Gr}_r(E)}(1)$ and $\phi^* L$, where ϕ is the projection in (2.2); this follows from the Seesaw Theorem [Mu, p. 54, Corollary 6]. Denote

$$\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}} \,:=\, \mathrm{NS}(\mathrm{Gr}_r(E)) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.3}$$

A cone in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ is a convex subset of it closed under the multiplication by nonnegative real numbers.

The effective cone of $\mathrm{Gr}_r(E)$ is the cone in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ (defined in (2.3)) generated by the effective divisors. The pseudo-effective cone of $\mathrm{Gr}_r(E)$ is the closure, in $\mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$, of the effective cone. The pseudo-effective cone of $\mathrm{Gr}_r(E)$ was computed in [BHP], which will be briefly recalled.

2.1. When the characteristic is zero.

First assume that the characteristic of k is zero.

Let

$$0 \,=\, E_0 \,\subset\, E_1 \,\subset\, \cdots \,\subset\, E_{m-1} \,\subset\, E_m \,=\, E$$

be the Harder–Narasimhan filtration of E [HL, p. 16, Theorem 1.3.4]. Let $1 \leq \ell \leq m$ be the unique integer such that $\mathrm{rank}(E_{\ell-1}) < r \leq \mathrm{rank}(E_\ell)$. Define

$$\lambda \,:=\, \mathrm{degree}(E_{\ell-1}) + (r - \mathrm{rank}(E_{\ell-1}))\mu(E_\ell/E_{\ell-1}), \tag{2.4}$$

where $\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)}$.

Lemma 2.1 ([BHP, p. 74, Theorem 4.1])**.** *The pseudo-effective cone of* $\mathrm{Gr}_r(E)$ *is generated by* $\phi^* c_1(L)$ *and* $c_1(\mathcal{O}_{\mathrm{Gr}_r(E)}(1)) - \lambda \phi^* c_1(L)$*, where* λ *is defined in* (2.4)*.*

2.2. When the characteristic is positive.

Assume that the characteristic of k is p, with $p > 0$.

For any vector bundle W on X, we have the vector bundle $F_X^* W$ on X, where F_X is the absolute Frobenius morphism of X. We recall that $F_X^* W$ is the subbundle of $W^{\otimes p}$ defined by the image of the morphism $W \longrightarrow W^{\otimes p}$ that sends any $w \in W$ to $w^{\otimes p}$. For any $j \geq 1$, the j–fold iteration of $W \longmapsto F_X^* W$ will be denoted by $(F_X^j)^* W$; by $(F_X^0)^* W$ we will denote W.

For any $j \geq 0$, let

$$0 \,=\, E_{j,0} \,\subset\, E_{j,1} \,\subset\, \cdots \,\subset\, E_{j,n_j} \,=\, (F_X^j)^* E$$

be the Harder–Narasimhan filtration of $(F_X^j)^* E$. There is a nonnegative integer $\delta = \delta(E)$ such that

$$0 \,=\, (F_X^j)^* E_{\delta,0} \,\subset\, (F_X^j)^* E_{\delta,1} \,\subset\, \cdots \,\subset\, (F_X^j)^* E_{\delta,n_\delta} \,=\, (F_X^j)^* (F_X^\delta)^* E \,=\, (F_X^{\delta+j})^* E$$

is the Harder–Narasimhan filtration of $(F_X^{\delta+j})^*E$ for all $j \geq 0$; so $n_\delta = n_{\delta+j}$ and $(F_X^j)^*E_{\delta,i} = E_{\delta+j,i}$ for all $1 \leq i \leq n_\delta$. Note that if δ satisfies the above condition, then any integer greater than δ also satisfies the above condition. Also, if δ satisfies the above condition, then $E_{\delta,i}/E_{\delta,i-1}$ is strongly semistable for all $1 \leq i \leq n_\delta$.

Fix a δ satisfying the above condition. Let $1 \leq \ell \leq n_\delta$ be the unique integer such that $\text{rank}(E_{\delta,\ell-1}) < r \leq \text{rank}(E_{\delta,\ell})$. Define

$$\lambda := \frac{1}{p^\delta}\left(\text{degree}(E_{\delta,\ell-1}) + (r - \text{rank}(E_{\delta,\ell-1}))\mu(E_{\delta,\ell}/E_{\delta,\ell-1})\right). \tag{2.5}$$

Note that λ does not depend on the choice of δ.

Lemma 2.2 ([BHP, p. 76, Theorem 4.4])**.** *The pseudo-effective cone of* $\text{Gr}_r(E)$ *is generated by* $\phi^*c_1(L)$ *and* $c_1(\mathcal{O}_{\text{Gr}_r(E)}(1)) - \lambda\phi^*c_1(L)$*, where* λ *is defined in* (2.5)*.*

3. The effective cone

Set $k = \overline{\mathbb{F}}_p$, with $p > 0$.

The following proposition describes the effective cone of $\text{Gr}_r(E)$ contained in the pseudo-effective cone of $\text{Gr}_r(E)$.

Theorem 3.1. *The effective cone of* $\text{Gr}_r(E)$ *coincides with the pseudo-effective cone of* $\text{Gr}_r(E)$*.*

Proof. The pseudo-effective cone of $\text{Gr}_r(E)$ is described in Lemma 2.2. We need to show that the two boundary edges are contained in the effective cone of $\text{Gr}_r(E)$.

The class $\phi^*c_1(L)$ in Lemma 2.2 is given by a fiber of the map ϕ in (2.2). So it suffices to show that the class $c_1(\mathcal{O}_{\text{Gr}_r(E)}(1)) - \lambda\phi^*c_1(L)$ in Lemma 2.2 lies in the effective cone.

Fix a pair

$$(Y, \Phi), \tag{3.1}$$

where Y is an irreducible smooth projective curve and

$$\Phi : Y \longrightarrow X$$

is a dominant morphism of such that there is a line bundle $\mathcal{L}_0$ on Y satisfying the condition that

$$\Phi^*E = \bigoplus_{i=1}^N \mathcal{L}_0^{\otimes a_i} \tag{3.2}$$

(see (2.1)), where a_i are integers; see [BP, p. 214, Proposition 2.1] for the existence of a pair as in (3.1), (3.2) (see also [Mo, p. 809, Theorem 2.2]).

Consider the vector bundle

$$(F_Y^\delta)^*\Phi^*E = \Phi^*(F_X^\delta)^*E$$

on Y (see (2.5)), where F_Y is the absolute Frobenius morphism of Y. Denoting $(F_Y^\delta)^*\mathcal{L}_0 = \mathcal{L}_0^{\otimes p^\delta}$ by $\mathcal{L}$, from (3.2) we have

$$\mathcal{E} \;:=\; \Phi^*(F_X^\delta)^*E \;=\; \bigoplus_{i=1}^N (F_Y^\delta)^*\mathcal{L}_0^{\otimes a_i} \;=\; \bigoplus_{i=1}^N \mathcal{L}^{\otimes a_i}. \tag{3.3}$$

Recall that $E_{\delta,j}/E_{\delta,j-1}$ is strongly semistable for all $1 \leq j \leq n_\delta$. This implies that $\Phi^*(E_{\delta,j}/E_{\delta,j-1}) = (\Phi^*E_{\delta,j})/(\Phi^*E_{\delta,j-1})$ is strongly semistable for every $1 \leq j \leq n_\delta$. Consequently,

$$0 \;=\; \Phi^*E_{\delta,0} \subset \Phi^*E_{\delta,1} \subset \cdots \subset \Phi^*E_{\delta,n_\delta} \;=\; \Phi^*(F_X^\delta)^*E \;=\; \mathcal{E} \tag{3.4}$$

is the Harder–Narasimhan filtration of $\mathcal{E}$ (defined in (3.3)). From (3.3) and (3.4) it follows that each subbundle $\Phi^*E_{\delta,j} \subset \mathcal{E}$ in (3.4) is a direct sum of some of the direct summands in (3.3). Let σ be a permutation of $\{1, \cdots, N\}$ such that

$$\Phi^*E_{\delta,j} \;=\; \bigoplus_{i=1}^{\mathrm{rank}(E_{\delta,j})} \mathcal{L}^{\otimes a_{\sigma(i)}}$$

for all $1 \leq j \leq n_\delta$. Hence

$$\widetilde{\mathcal{L}} \;:=\; \left(\bigotimes_{i=1}^n \mathcal{L}^{\otimes a_{\sigma(i)}}\right) \otimes \left(\mathcal{L}^{\otimes a_{\sigma(n+1)}}\right)^{\otimes(r-n)} \tag{3.5}$$

is a direct summand of

$$\bigwedge^r \mathcal{E} \;=\; \bigwedge^r \Phi^*(F_X^\delta)^*E,$$

where $n = \mathrm{rank}(E_{\delta,\ell-1})$ (see (2.5) for ℓ). This implies that

$$\mathcal{O}_Y \;\subset\; \left(\bigwedge^r \mathcal{E}\right) \otimes \left(\widetilde{\mathcal{L}}\right)^*, \tag{3.6}$$

where $\widetilde{\mathcal{L}}$ is the line bundle in (3.5) and $\mathcal{E}$ is the vector bundle in (3.3).

From (3.3) and (3.4) we deduce that

$$\mathrm{degree}\left(\widetilde{\mathcal{L}}\right) \;=\; \lambda \cdot \mathrm{degree}(\Phi \circ F_Y^\delta) \;=\; \lambda \cdot \mathrm{degree}(F_X^\delta \circ \Phi), \tag{3.7}$$

where λ is defined in (2.5).

Let

$$\varphi \;:\; \mathrm{Gr}_r(\mathcal{E}) \;\longrightarrow\; Y \tag{3.8}$$

be the Grassmann bundle parametrizing the r dimensional quotients of the fibers of the vector bundle $\mathcal{E}$ in (3.3). Let

$$\mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1) \;\longrightarrow\; \mathrm{Gr}_r(\mathcal{E})$$

be the tautological line bundle whose fiber over the point of $\mathrm{Gr}_r(E)$ representing a quotient $\mathcal{E}_y \longrightarrow Q$, where $y \in Y$, is $\bigwedge^r Q$. We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Gr}_r(\mathcal{E}) & \xrightarrow{\Psi} & \mathrm{Gr}_r(E) \\ \Big\downarrow \varphi & & \Big\downarrow \phi \\ Y & \xrightarrow{F_X^\delta \circ \Phi} & X \end{array} \tag{3.9}$$

(see (3.8), (2.2) and (3.1)). We have

$$\Psi^*\mathcal{O}_{\mathrm{Gr}_r(E)}(1) \;=\; \mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1), \tag{3.10}$$

where Ψ is the map in (3.9).

Combining (3.7) and (3.10) it follows that

$$\Psi^*(c_1(\mathcal{O}_{\mathrm{Gr}_r(E)}(1)) - \lambda\phi^* c_1(L)) \;=\; c_1(\mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1)) - \varphi^* c_1(\widetilde{\mathcal{L}}), \tag{3.11}$$

where $\widetilde{\mathcal{L}}$ is the line bundle in (3.5). By the projection formula,

$$H^0\left(\mathrm{Gr}_r(\mathcal{E}),\, \mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1)\otimes\varphi^*\left(\widetilde{\mathcal{L}}\right)^*\right) \;=\; H^0\left(Y,\, \left(\bigwedge\nolimits^r \mathcal{E}\right)\otimes\left(\widetilde{\mathcal{L}}\right)^*\right),$$

and hence from (3.6) we conclude that $H^0\left(\mathrm{Gr}_r(\mathcal{E}),\, \mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1)\otimes\varphi^*\left(\widetilde{\mathcal{L}}\right)^*\right) \neq 0$. This implies that the Néron–Severi class of $\mathcal{O}_{\mathrm{Gr}_r(\mathcal{E})}(1)\otimes\varphi^*\left(\widetilde{\mathcal{L}}\right)^*$ is effective.

Note that the pull-back map $\Psi^* : \mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}} \longrightarrow \mathrm{NS}(\mathrm{Gr}_r(\mathcal{E}))_{\mathbb{R}}$ induced by Ψ is an isomorphism. Therefore, from (3.11) it follows that the class $c_1(\mathcal{O}_{\mathrm{Gr}_r(E)}(1)) - \lambda\phi^* c_1(L) \in \mathrm{NS}(\mathrm{Gr}_r(E))_{\mathbb{R}}$ is effective. This completes the proof. $\square$

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