

1 Derivations and Differentials

2 Contangent complexes and local properties of morphisms

The lectures 1 and 2 were given on 18/09/2026 and 21/09/2026. The following are the note for the lecture 3 to be given on 22/09/2026. Corrections/suggestions are welcome.

3 Lifting problem for algebras

3.1 The guiding philosophy Given a morphism $X \rightarrow Y$ of schemes and a quasi-coherent $\mathcal{O}_X$ -module M , we get the groupoid $\mathfrak{E}xt_Y(X, M)$ of M -extensions of X over Y . There is a functorial binary operation of addition of such extensions, which makes $\mathfrak{E}xt_Y(X, M)$ an “abelian-group-like category” (more precisely, a Picard category). The groupoid $\mathfrak{E}xt_Y(X, M)$ can be calculated using the cotangent complex $L_{X/Y}$, which is a complex of flat $\mathcal{O}_X$ -modules in $C^{\leq 0}(\mathcal{O}_X)$. In terms of the cotangent complex, the set (which, it so happens, is naturally an abelian group) $Exal_Y(X, M)$ of all isomorphism classes of objects of $\mathfrak{E}xt_Y(X, M)$ is identified with $Ext^1(L_{X/Y}, M)$, and for each object, the automorphism group $Hom(\Omega_{X/Y}, M)$ is identified with $Ext^0(L_{X/Y}, M)$.

The following is a helpful analogy. The category $\mathfrak{Pic}(X)$ of line bundles on X has a functorial tensor product on it with natural associativity, commutativity, unit and inverse in functorial terms, making it a Picard category (this is the origin of the concept of a Picard category). We can calculate the groupoid $\mathfrak{Pic}(X)$ using the sheaf $\mathcal{O}_X^\times$, as follows. The set (which is naturally an abelian group) $Pic(X)$ of all isomorphism classes of objects of $\mathfrak{Pic}(X)$ is identified with $H^1(X, \mathcal{O}_X^\times)$, and for each object (means a line bundle L), the automorphism group $Aut(L) = \mathcal{O}_X(X)^\times$ is identified with $H^0(X, \mathcal{O}_X^\times)$.

Recall that in various situations, the groups $Pic(X)$ and $\mathcal{O}_X(X)^\times$, because of their cohomological interpretation, can be determined via the truncated complex $0 \rightarrow I^0 \rightarrow Z(I^1) \rightarrow 0 = \tau_{\leq 1} I^\bullet$ where $\mathcal{O}_X^\times \rightarrow I^\bullet$ is a quasi-isomorphism to a complex $I^\bullet$ of injective sheaves of abelian groups on X , as the groups $Ext^1(\mathbb{Z}, \tau_{\leq 1} I^\bullet) = Ext^1(\mathbb{Z}, I^\bullet)$ and $Ext^0(\mathbb{Z}, \tau_{\leq 1} I^\bullet) = Ext^0(\mathbb{Z}, I^\bullet)$. Similarly, the groups $Exal_Y(X, M)$ and $Hom(\Omega_{X/Y}, M)$ can be determined using the truncated cotangent complex $\tau_{\geq -1} L_{X/Y} \in D^{[-1, 0]}(\mathcal{O}_X)$ as the groups $Ext^1(\tau_{\geq -1} L_{X/Y}, M) = Ext^1(L_{X/Y}, M)$ and $Ext^0(\tau_{\geq -1} L_{X/Y}, M) = Ext^0(L_{X/Y}, M)$. The point is that the lower cohomologies are already captured faithfully by the truncations, and you do not need to use the full complexes for them.

In discussing the lifting problem for schemes in this section, we will restrict ourselves to the category of affine schemes, where the theory is simpler and is directly

expressible in commutative algebraic terms. Hence, we will talk about the lifting problem for commutative algebras.

Exal

3.2 If $A \xrightarrow{f} B \xrightarrow{g} C$ are ring homomorphisms (we will denote the induced morphisms of affine schemes by the same letters $\text{Spec } C \xrightarrow{g} \text{Spec } B \xrightarrow{f} \text{Spec } A$ for simplicity, as there is no danger of confusion) we get a functorial exact sequence of C -modules

$$\Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$$

Let N be a C -module (which we regard as a sheaf of quasi-coherent $\mathcal{O}_{\text{Spec } C}$ -modules, and again denote it simply by N), and let g_*N denote the B -module which is its direct image under $g : \text{Spec } C \rightarrow \text{Spec } B$. Note that $\text{Der}_A(B, g_*N)$ is naturally a C -module, and we have natural isomorphisms

$$\text{Der}_A(B, g_*N) = \text{Hom}_{B\text{-mod}}(\Omega_{B/A}, g_*N) = \text{Hom}_{C\text{-mod}}(\Omega_{B/A} \otimes_B C, N).$$

Hence applying $\text{Hom}_{C\text{-mod}}(-, N)$ to the right exact sequence $\Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$ we get the following left exact sequence of C -modules.

$$0 \rightarrow \text{Der}_B(C, N) \xrightarrow{g} \text{Der}_A(C, N) \xrightarrow{f} \text{Der}_A(B, g_*N)$$

3.3 The groupoid $\mathfrak{E}xal_A(B, M)$ and the set $Exal_A(B, M)$. If $f : A \rightarrow B$ is a ring homomorphism and M a B -module, $\mathfrak{E}xal_A(B, M)$ will denote the category of all **small A -algebra extensions of B by M** ('small' means square-zero). An object in this category is a tuple $(A \rightarrow B', q : B' \rightarrow B, \phi : \ker(q) \rightarrow M)$ where $A \rightarrow B'$ is an A -algebra, $q : B' \rightarrow B$ is a surjective A -algebra homomorphism such that $\ker(q)^2 = 0$, and $\phi : \ker(q) \rightarrow M$ is a B -linear isomorphism of modules, where $\ker(q)$ is naturally a B -module as $\ker(q)^2 = 0$. By abuse of notation, we will denote such a tuple simply by $0 \rightarrow M \rightarrow B' \rightarrow B \rightarrow 0$.

A morphism in the category $\mathfrak{E}xal_A(B, M)$ from $0 \rightarrow M \rightarrow B' \rightarrow B \rightarrow 0$ to $0 \rightarrow M \rightarrow B'' \rightarrow B \rightarrow 0$ is an A -algebra homomorphism $\psi : B' \rightarrow B''$ which makes the following diagram commute.

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & B' & \rightarrow & B \rightarrow 0 \\ & & \parallel & & \psi \downarrow & & \parallel \\ 0 & \rightarrow & M & \rightarrow & B'' & \rightarrow & B \rightarrow 0 \end{array}$$

Note that by 5-lemma $\psi : B' \rightarrow B''$ is necessarily an isomorphism, so $\mathfrak{E}xal_A(B, M)$ is a groupoid. We denote by $Exal_A(B, M)$ the set of isomorphism classes in $\mathfrak{E}xal_A(B, M)$ (all the isomorphism classes in $\mathfrak{E}xal_A(B, M)$ indeed form a set, as a representative ring for any such B' can always be chosen to have the set $B \times M$ its underlying set).

3.4 The zero objects in $\mathfrak{E}al_A(B, M)$. Let $f : A \rightarrow B$ be a ring homomorphism and let M be a B -module. There is a distinguished object $0 \rightarrow M \xrightarrow{i} B\langle M \rangle \xrightarrow{q} B \rightarrow 0$ in $\mathfrak{E}al_A(B, M)$ defined as follows. As a ring, $B\langle M \rangle$ has underlying abelian group $B \oplus M$, with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. The A -algebra structure $A \rightarrow B\langle M \rangle$ is given by $a \mapsto (f(a), 0)$. The map $q : B\langle M \rangle \rightarrow B$ is given by the first projection, and $i : M \rightarrow B\langle M \rangle$ by $m \mapsto (0, m)$. For simplicity, we write this object of $\mathfrak{E}al_A(B, M)$ simply as $B\langle M \rangle$.

Notation. The object $B\langle M \rangle$ is denoted by $B \ltimes M$ in some other sections of these lecture notes.

Exercise. An object $0 \rightarrow M \xrightarrow{j} B' \xrightarrow{p} B \rightarrow 0$ in $\mathfrak{E}al_A(B, M)$ is isomorphic to $B\langle M \rangle$ if and only if $p : B' \rightarrow B$ admits a section $\sigma : B \rightarrow B'$ in A -algebras.

3.5 Automorphisms in $\mathfrak{E}al_A(B, M)$ are given by $Der_A(B, M)$. Given any object $X = (0 \rightarrow M \xrightarrow{i} B' \xrightarrow{q} B \rightarrow 0)$ in $\mathfrak{E}al_A(B, M)$, and any $v : B \rightarrow M$ in $Der_A(B, M)$, we get an automorphism of X defined by the map $B' \rightarrow B' : b' \mapsto b' + ivq(b')$. This gives a bijection $Der_A(B, M) \rightarrow Aut(X)$ of sets, which is in fact an isomorphism of B -modules (the natural B -module structure on the automorphism group is described below).

3.6 The B -linear Picard category structure. In statement 3.9 we will functorially associate with any pair of objects X and Y another object $X + Y$, which defines a functor $+ : \mathfrak{E}al_A(B, M) \times \mathfrak{E}al_A(B, M) \rightarrow \mathfrak{E}al_A(B, M)$. The trivial extension $0 \rightarrow M \rightarrow B\langle M \rangle \rightarrow B \rightarrow 0$, where $B\langle M \rangle \rightarrow B$ is the projection $p_1 : B \oplus M \rightarrow B$, together with A -algebra structure given by $(f, 0) : A \rightarrow B\langle M \rangle$, is an object $O_{A, B, M}$ of $\mathfrak{E}al_A(B, M)$. To any object $X = (0 \rightarrow M \xrightarrow{i} B' \xrightarrow{p} B \rightarrow 0)$ we can functorially associate another object $-X = (0 \rightarrow M \xrightarrow{-i} B' \xrightarrow{p} B \rightarrow 0)$ (where i is replaced by $-i$). With any object X and any element $\lambda \in B$, we associate (see statement 3.9) an object λX . The addition functor $+$, zero object $O_{A, B, M}$, additive inverse $X \mapsto -X$ and scalar multiplication λX described above, together with appropriate functorial versions of associativity, commutativity, and additive identity, make $\mathfrak{E}al_A(B, M)$ a Picard groupoid of B -modules, with the B -module of automorphisms of any object being $Der_A(B, M)$.

3.7 Push-forward functor $\phi_* : \mathfrak{E}al_A(B, I) \rightarrow \mathfrak{E}al_A(B, J)$. Let $f : A \rightarrow B$ be a ring homomorphism and $\phi : I \rightarrow J$ a homomorphism of B -modules. Given an object $X = (0 \rightarrow I \xrightarrow{i} B' \xrightarrow{p} B \rightarrow 0)$ in $\mathfrak{E}al_A(B, I)$, let (B'', j, ψ) be the push-out A -module where

$$B'' = \frac{B' \oplus J}{(i, -\phi)(I)}$$

together with maps $j : J \hookrightarrow B'' : y \mapsto (0, y)$ and $\psi : B' \rightarrow B'' : b' \mapsto (b', 0)$. We define a multiplication on B'' by putting $(b'_1, y_1) \times (b'_2, y_2) = (b'_1 b'_2, b'_1 y_2 + b'_2 y_1)$. We define an A -algebra structure on B'' by the ring homomorphism $A \rightarrow B'' :$

$a \mapsto (f'(a), 0)$ where the ring homomorphism $f' : A \rightarrow B'$ defines the A -algebra structure on B' . The map $q : B'' \rightarrow B$ be defined by $(b', y) \mapsto p(b')$ is an A -algebra homomorphism with kernel (isomorphic via j to) J , with $J^2 = 0$, so we get a small extension $\phi_*(X)$. This gives a functor $\phi_* : \mathfrak{E}xal_A(B, I) \rightarrow \mathfrak{E}xal_A(B, J)$, and so a set map $\phi_* : Exal_A(B, I) \rightarrow Exal_A(B, J)$ (after we define the B -module structure on $Exal_A(B, M)$ in statement 3.9, it will be clear that this map is B -linear). Note that the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \rightarrow & I & \xrightarrow{i} & B' & \xrightarrow{p} & B \rightarrow 0 \\ & & \phi \downarrow & & \psi \downarrow & & \parallel \\ 0 & \rightarrow & J & \xrightarrow{j} & B'' & \xrightarrow{q} & B \rightarrow 0 \end{array}$$

Moreover, sending $\phi \mapsto \phi_*$ makes $Exal_A(B, -) : M \mapsto Exal_A(B, M)$ a functor from B -modules to B -modules.

3.8 Pull-back $g^* : Exal_A(C, M) \rightarrow Exal_A(B, g_*M)$. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. Let M be a C -module, and g_*M denote the resulting B -module M , where B operates via g . Given $0 \rightarrow M \xrightarrow{i} C' \xrightarrow{p} C \rightarrow 0$ in $Exal_A(C, M)$, let $B' = C' \times_{p, C, g} B$ be the fiber product A -algebra with projections $q : B' \rightarrow B$ and $g' : B' \rightarrow C'$. The kernel of q is again I with $I^2 = 0$, so we get a small extension of B by g_*M , giving a well-defined map $g^* : Exal_A(C, M) \rightarrow Exal_A(B, g_*M)$. The following diagram commutes.

$$\begin{array}{ccccccc} 0 & \rightarrow & g_*M & \rightarrow & B' & \xrightarrow{q} & B \rightarrow 0 \\ & & \parallel & & g' \downarrow & & g \downarrow \\ 0 & \rightarrow & M & \xrightarrow{i} & C' & \xrightarrow{p} & C \rightarrow 0 \end{array}$$

Once we define the module structure on $Exal$ in statement 3.9, it can be seen that g^* is B -linear when $Exal_A(C, M)$ (which is naturally a C -module) is made a B -module via $g : B \rightarrow C$, that is, $g^* : g_*Exal_A(C, M) \rightarrow Exal_A(B, g_*M)$ is B -linear.

3.9 There is a natural B -module structure on $Exal_A(B, M)$ defined as follows. Given any two objects of $\mathfrak{E}xal_A(B, M)$, say $X = (0 \rightarrow M \xrightarrow{i'} B' \xrightarrow{p'} B \rightarrow 0)$ and $Y = (0 \rightarrow M \xrightarrow{i''} B'' \xrightarrow{p''} B \rightarrow 0)$, with A -algebra structures $f' : A \rightarrow B'$ and $f'' : A \rightarrow B''$, we get an object

$$Z = (0 \rightarrow M \oplus M \xrightarrow{(i', i'')} B' \times_{p', B, p''} B'' \rightarrow B \rightarrow 0)$$

of $\mathfrak{E}xal_A(B, M \oplus M)$, with A -algebra structure on the ring $B' \times_{p', B, p''} B''$ defined by the map $(f', f'') : A \rightarrow B' \times_{p', B, p''} B''$.

Let $\alpha : M \oplus M \rightarrow M$ be the addition map $(n', n'') \mapsto n' + n''$. By the push-forward construction (see statement 3.7), we get an object $\alpha_*(Z)$ in $\mathfrak{E}xal_A(B, M)$.

We define a functor $+$: $\mathfrak{E}xal_A(B, M) \times \mathfrak{E}xal_A(B, M) \rightarrow \mathfrak{E}xal_A(B, M)$ by putting $X + Y = \alpha_*(Z)$. Concretely, $X + Y$ has the form $0 \rightarrow M \rightarrow R \rightarrow B \rightarrow 0$ where

$$R = \frac{(B' \times_{p', B, p''} B'') \oplus M}{(i', i'', -\alpha)(M \oplus M)}$$

The multiplication on R is defined by putting

$$(b'_1, b''_1, n_1) \times (b'_2, b''_2, n_2) = (b'_1 b'_2, b''_1 b''_2, b_1 n_2 + b_2 n_1)$$

where $(b'_1, b''_1) \mapsto b_1$ and $(b'_2, b''_2) \mapsto b_2$ under the map $B' \times_B B'' \rightarrow B$. The ring homomorphism $A \rightarrow R$ is defined by $a \mapsto (f'(a), f''(a), 0)$. By defining the sum of the isomorphism classes of X and Y to be the isomorphism class of $\alpha_*(Z)$, we get an additive group structure on the set $Exal_A(B, M)$. Scalar multiplication by $\lambda \in B$ gives a linear map $\lambda : M \rightarrow M$. We define the scalar multiplication functor λ on $\mathfrak{E}xal_A(B, M)$ to be the corresponding push-forward functor $\lambda_* : \mathfrak{E}xal_A(B, M) \rightarrow \mathfrak{E}xal_A(B, M)$ given in statement 3.7. This definition of scalar multiplication functor makes the set of isomorphism classes $Exal_A(B, M)$ a B -module.

3.10 The boundary map ∂ . Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms, and let N be a C -module. To any A -derivation $v : B \rightarrow g_* N$ (that is, $v \in Der_A(B, g_* N)$) we will associate the B -algebra extension $0 \rightarrow N \rightarrow C' \rightarrow C \rightarrow 0$ where as a ring, $C' = C\langle N \rangle$, the map $C' \rightarrow C$ is the projection $C \oplus N \rightarrow C$ which has kernel N , and the B -algebra structure on C' is given by the ring homomorphism

$$(g, v) : B \rightarrow C \oplus N : b \mapsto (g(b), v(b))$$

Let $\partial(v) \in Exal_B(C, N)$ denote the isomorphism class of the above small extension. This defines a C -linear map

$$\partial : Der_A(B, g_* N) \rightarrow Exal_B(C, N)$$

which is functorial in $A \xrightarrow{f} B \xrightarrow{g} C$ and N . With this, the following sequence of C -modules is exact.

$$\begin{aligned} 0 \rightarrow Der_B(C, N) \rightarrow Der_A(C, N) &\xrightarrow{g^*} Der_A(B, g_* N) \xrightarrow{\partial} \\ Exal_B(C, N) \rightarrow Exal_A(C, N) &\xrightarrow{g^*} Exal_A(B, g_* N) \end{aligned}$$

3.11 The injective map $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$. Any element of $Ext_B^1(\Omega_{B/A}, M)$ is represented by a short-exact sequence $0 \rightarrow M \xrightarrow{i} E \xrightarrow{j} \Omega_{B/A} \rightarrow 0$. Let $d : B \rightarrow \Omega_{B/A}$ be the universal A -linear derivation. Consider the fiber product A -module

$$B' = B \times_{d, \Omega_{B/A}, j} E$$

which fits in the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow M & \xrightarrow{k} & B' & & \xrightarrow{\pi_1} & B & \rightarrow 0 \\ & & \parallel & & \pi_2 \downarrow & d \downarrow & \\ 0 \rightarrow M & \xrightarrow{i} & E & \xrightarrow{j} & \Omega_{B/A} & \rightarrow 0 \end{array}$$

where $\pi_1 : B' \rightarrow B$ and $\pi_2 : B' \rightarrow E$ are the two projections, and $k : M \rightarrow B'$ is defined by $x \mapsto (0, i(x))$.

We define a multiplication on B' by putting $(b_1, x_1) \times (b_2, x_2) = (b_1 b_2, b_1 x_2 + b_2 x_1)$, which makes B' a ring. The A -algebra structure is given by the homomorphism $A \rightarrow B' : a \mapsto (f(a), 0)$ where $f : A \rightarrow B$ is the structure homomorphism for B . The projection $B' \rightarrow B$ is an A -algebra homomorphism, with kernel M , which has square zero. This gives a well-defined B -linear homomorphism $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$.

The B -module E becomes a B' -module via $B' \rightarrow B$. With this, the map $\pi_2 : B' \rightarrow E$ is an A -derivation. This induces a B' -linear map $\alpha : \Omega_{B'/A} \rightarrow E$, which takes the universal derivation $d' : B' \rightarrow \Omega_{B'/A}$ to the derivation $\pi_2 : B' \rightarrow E$.

Corresponding to the ring homomorphisms $A \rightarrow B' \rightarrow B$, we have an exact sequence of B -modules (the ‘second exact sequence’ of Kähler differentials)

$$M \xrightarrow{\delta} \Omega_{B'/A} \otimes_{B'} B \rightarrow \Omega_{B/A} \rightarrow 0$$

The map $\delta : M \rightarrow \Omega_{B'/A} \otimes_{B'} B : y \mapsto d'(0, y)$ is injective, as by definition of α , the following diagram commutes, and i is injective.

$$\begin{array}{ccccc} M & \xrightarrow{\delta} & \Omega_{B'/A} \otimes_{B'} B & \rightarrow & \Omega_{B/A} \rightarrow 0 \\ \parallel & & \alpha \downarrow & & \parallel \\ 0 \rightarrow M & \xrightarrow{i} & E & \xrightarrow{j} & \Omega_{B/A} \rightarrow 0 \end{array}$$

Now by 5-lemma, $\alpha : \Omega_{B'/A} \rightarrow E$ is an isomorphism of B -modules. This shows that the extension class of $(0 \rightarrow M \rightarrow E \rightarrow \Omega_{B/A} \rightarrow 0)$ in $Ext_B^1(\Omega_{B/A}, M)$ can be recovered by from the class of the corresponding algebra extension $(B' \rightarrow B)$ in $Exal_A(B, M)$ as the second Kähler differential sequence for $A \rightarrow B' \rightarrow B$, which is short exact. Hence the map $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$ is injective.

3.12 The image of $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$. The above shows that the image of $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$ consists of all isomorphism classes of square-zero A -algebra extensions $0 \rightarrow M \xrightarrow{u} C \xrightarrow{q} B \rightarrow 0$ such that the corresponding map $\delta : M \rightarrow \Omega_{C/A} \otimes_C B$ is injective, giving a short exact sequence $0 \rightarrow M \rightarrow \Omega_{C/A} \otimes_C B \rightarrow \Omega_{B/A} \rightarrow 0$ as the second Kähler differential sequence for $A \rightarrow C \rightarrow B$. In the course of the construction of the extension B' from an element of $Ext_B^1(\Omega_{B/A}, M)$, we saw that the map δ is injective. Conversely, consider a square-zero A -algebra extension $0 \rightarrow M \xrightarrow{u} C \xrightarrow{q} B \rightarrow 0$ such that the corresponding map $\delta : M \rightarrow \Omega_{C/A} \otimes_C B$ is injective. This means the second fundamental sequence gives a short exact $0 \rightarrow M \rightarrow \Omega_{C/A} \otimes_C B \rightarrow \Omega_{B/A} \rightarrow 0$ of B -modules, which defines an element of $Ext_B^1(\Omega_{B/A}, M)$. The above construction of an algebra extension from a module extension will now recover the algebra extension $0 \rightarrow M \xrightarrow{u} C \xrightarrow{q} B \rightarrow 0$. Hence, C is in the image of $Ext_B^1(\Omega_{B/A}, M) \rightarrow Exal_A(B, M)$, as claimed.

3.13 Formulation of the first-order infinitesimal lifting problem for algebras. Let A' be a commutative ring, $J \subset A'$ a finite ideal with $J^2 = 0$ and $q : A' \rightarrow A' / J = A$ the quotient. Let $f : A \rightarrow B$ be a homomorphism of commutative rings, where B is noetherian. What does it mean to lift B to A' ? The meaning can be made precise in the following steps.

Triples (B', f', p) . To begin with, we want to have triples (B', f', p) where B' is a commutative ring together with ring homomorphisms $f' : A' \rightarrow B'$ and $p : B' \rightarrow B$ such that $p \circ f' = f \circ q : A' \rightarrow B$. On such triples, we will impose two further conditions (1) and (2).

(1) Restriction condition. The induced map $(f, p) : A \otimes_{q, A', f'} B' \rightarrow B$ is an isomorphism of A' -algebras.

Consequences of (1): If (B', f', p) satisfies the condition (1), then p is surjective, $\ker(p) = f'(J)B'$, $B = B' / f'(J)B'$, $\ker(p)^2 = 0$ and $J \otimes_{A'} B' = J \otimes_{A'} B = J \otimes_A B$.

Note: The condition (1) is trivially satisfied by the triple $B' = B$, $f' = f \circ q$ and $p = \text{id}_B$, in particular, $f'(J) = 0$, so $f'(J)B' = 0$.

(2) Proto-flatness condition. The map $p : B' \rightarrow B$ is surjective, $\ker(p)^2 = 0$ (so that $\ker(p)$ is naturally a B -module), and the underlying B -module of $\ker(p)$ is isomorphic to $J \otimes_A B$.

Note. The condition (2) demands that there should exist some arbitrary B -module isomorphism ϕ of $\ker(p)$ with $J \otimes_A B$. It **does not demand** that the composition of $\phi : \ker(p) \rightarrow J \otimes_A B$ with the natural map $m : J \otimes_A B \rightarrow B'$ (defined below) should equal the given inclusion $\ker(p) \hookrightarrow B'$. But it will turn out that the condition (2) will imply that there will indeed exist a unique new ϕ which takes the inclusion $\ker(p) \hookrightarrow B'$ to m , in particular, m will be injective if (2) is satisfied.

3.14 Element of $\text{Exal}_{A'}(B, J \otimes_A B)$ canonically associated to a lift (B', f', p) . We will show that if (B', f', p) is a lift of f along q that satisfies (1) and (2), and if J is a finite ideal in A' and if B is noetherian, then we get an exact sequence

$$\mathcal{E} = (0 \rightarrow J \otimes_A B \xrightarrow{m} B' \xrightarrow{p} B \rightarrow 0)$$

where the map $m : J \otimes_A B \rightarrow B'$ is given by the formula

$$m(x \otimes_A p(b')) = f'(x)b'$$

for any $x \in J$ and $b' \in B'$. This is an object in $\mathfrak{E}xal_{A'}(B, J \otimes_A B)$, canonically associated to the lift (B', f', p) . Conversely, if $\mathcal{E} = (0 \rightarrow J \otimes_A B \xrightarrow{m} B' \xrightarrow{p} B \rightarrow 0)$ is an object of $\mathfrak{E}xal_{A'}(B, J \otimes_A B)$, in which $m : J \otimes_A B \rightarrow B'$ is given by the above formula, then the induced map $(p, f) : B' \otimes_{A'} A \rightarrow B$ is indeed an isomorphism, so (1) and (2) are satisfied. Hence, the set of all lifts that satisfy (1) and (2) is in a natural bijection with the subset of $\text{Exal}_{A'}(B, J \otimes_A B)$ corresponding to extensions of the form $0 \rightarrow J \otimes_A B \xrightarrow{m} B' \xrightarrow{p} B \rightarrow 0$, where m is given by the above formula.

3.15 Motivation. In the most important special case where B is flat over A , we moreover desire that B' should be flat over A' . In that case, applying the exact functor $- \otimes_{A'} B'$ to the short exact sequence $\mathbf{e} = (0 \rightarrow J \rightarrow A' \rightarrow A \rightarrow 0)$ of A' -modules will give us an exact sequence

$$0 \rightarrow J \otimes_A B \xrightarrow{m} B' \rightarrow A \otimes_{A'} B' \rightarrow 0.$$

In the above, the map $m : J \otimes_A B \rightarrow B'$ is given by $x \otimes_A p(b') \mapsto f'(x)b'$ for any $x \in J$ and $b' \in B'$. It is well defined as $f'(J) \subset \ker(p)$ and $\ker(p)^2 = 0$. Hence, we should be looking for triples $(B', f' : A' \rightarrow B', p : B' \rightarrow B)$ which satisfy (1), and moreover for which the resulting map $m : J \otimes_{A'} B \rightarrow B'$ is a kernel of p . But instead of demanding that m is a kernel of p , we will only impose the apparently weaker condition (2), which says that the kernel of p is isomorphic to $J \otimes_A B$ as a B -module.

Suppose that a triple (B', f', p) satisfies (2). Choose any B -module isomorphism $k : J \otimes_A B \rightarrow \ker(p)$ (there could be many choices of k , and the set of all such k is a principal $\text{Aut}_{B\text{-mod}}(J \otimes_A B)$ -set). This defines an object

$$\mathbf{E} = (0 \rightarrow J \otimes_A B \xrightarrow{k} B' \xrightarrow{p} B \rightarrow 0)$$

of $\mathfrak{E}xal_{A'}(B, J \otimes_A B)$ which depends on k . Here, the A' -algebra structure on B is via the composite $f \circ q : A' \rightarrow B$. Also note that as the object $\mathbf{E}$ is an A' -algebra extension, a specific homomorphism $f' : A' \rightarrow B'$ is implicit in $\mathbf{E}$, though the notation (in the form of the above short exact sequence) does not mention f' . Conversely, any object $\mathbf{E}$ of $\mathfrak{E}xal_{A'}(B, J \otimes_A B)$, on forgetting k , gives a triple (B', f', p) that satisfies (2). Thus, the condition (2) connects the lifting problem with the machinery of $Exal$ and so with the powerful calculus of cotangent complexes.

Clearly, (1) and (2) are satisfied by a flat lifting (B', f', p) of a flat algebra $f : A \rightarrow B$. Conversely, the reason (1) and (2) are sufficient for a flat lifting of a flat algebra $f : A \rightarrow B$ in the **noetherian case** (means A' and B are noetherian) is that together they imply the flatness of B'/A' . More precisely, let B be noetherian and $J \subset A'$ be a finite ideal. If a triple (B', f', p) satisfies the restriction condition and the proto-flatness condition, then $m : J \otimes_{A'} B \rightarrow B'$ is automatically a kernel of p by statement 3.17 below. From this the flatness of B/A follows from the easy case of the local criterion for flatness – see statement 3.20 below.

Hence we have the following strategy for checking the existence of a lift (B', f', p) of f under q . If we begin with an arbitrary element $\mathbf{E} = (0 \rightarrow J \otimes_A B \xrightarrow{k} B' \xrightarrow{p} B \rightarrow 0)$ of $Exal_{A'}(B, J \otimes_A B)$, then what more is required to get a lift? We just have to demand that p induces an isomorphism $(f, p) : A \otimes_{A'} B' \rightarrow B$ of A' -algebras, or equivalently by statement 3.17, that $m : 0 \rightarrow J \otimes_A B \rightarrow B'$ is injective. This requirement carves out a certain subset of $Exal_{A'}(B, J \otimes_A B)$, which is a principal set under $Exal_A(B, J \otimes_A B) = Ext^1(L_{B/A}, J \otimes_A B)$, possibly empty. This is non-empty if and only if a certain element $obs_{f,q} \in Ext^2(L_{B/A}, J \otimes_A B)$ is zero. The element $obs_{f,q}$ is known as the **obstruction class for lifting f along q** .

All the above is eventually shown below. Note that we no more require that B is flat over A .

3.16 Automorphisms of a lift (B', f', p) . Let (B', f', p) satisfy both (1) and (2). Let $\psi : B' \rightarrow B'$ be an automorphism of (B', f', p) , which means a ring automorphism $\psi : B' \rightarrow B'$ with $\psi \circ f' = f'$ and $p = p \circ \psi$. Any element of $\ker(p) = f'(J)B'$ is the sum of elements of the form $f'(x)b'$ where $x \in J$ and $b' \in B'$. As $p = p \circ \psi$, we have $p(b' - \psi(b')) = 0$, so $b' - \psi(b') \in \ker(p)$, and so $f'(x)(b' - \psi(b')) = 0$ as $\ker(p)^2 = 0$, hence $f'(x)b' = f'(x)\psi(b')$. As $\psi \circ f' = f'$, we have $\psi(f'(x)b') = \psi(f'(x))\psi(b') = f'(x)\psi(b')$. The previous two equations $f'(x)b' = f'(x)\psi(b')$ and $\psi(f'(x)b') = f'(x)\psi(b')$ imply that $f'(x)b' = \psi(f'(x)b')$. Hence ψ restricts to identity on $\ker(p)$. As $p = p \circ \psi$, it now follows that ψ is the same as an automorphism of the object $\mathbf{E} = (0 \rightarrow J \otimes B \xrightarrow{m} B' \xrightarrow{p} B \rightarrow 0)$ of $\mathfrak{E}ra_{A'}(B, J \otimes B)$, hence it is an element of $Der_{A'}(B, J \otimes B) = Der_A(B, J \otimes B)$. Thus, all automorphisms of a lift (B', f', p) form the group $Der_A(B, J \otimes B) = Hom(L_{B/A}, J \otimes B)$.

3.17 Restriction and kernel. Let A' be a ring, $J \subset A'$ a finite ideal with $J^2 = 0$, $q : A' \rightarrow A' / J = A$ the quotient, and let $f : A \rightarrow B$ be a ring homomorphism where B is noetherian. Let B be regarded as an A' -algebra via the composite $h = f \circ q : A' \rightarrow B$. Let $f' : A' \rightarrow B'$ be an A' -algebra, and let $p : B' \rightarrow B$ be a surjective A' -algebra homomorphism such that $\ker(p)^2 = 0$, and moreover assume that there exists some B -module isomorphism $J \otimes_A B \cong \ker(p)$. Consider the following diagram in which the rows are exact, and the square is commutative.

$$\begin{array}{ccccccc} 0 \rightarrow & J \otimes_A B & \xrightarrow{k} & B' & \xrightarrow{p} & B & \rightarrow 0 \\ & & & \uparrow f' & & \uparrow f & \\ 0 \rightarrow & J & \rightarrow & A' & \xrightarrow{q} & A & \rightarrow 0 \end{array}$$

Then the following two statements are equivalent.

(1) The induced map

$$A \otimes_{q, A', f'} B' \xrightarrow{(f, p)} B$$

is an isomorphism of A' -algebras.

(2) Let $m : J \otimes_A B \rightarrow B'$ be the A' -module homomorphism defined by

$$\sum x_i \otimes p(y_i) \mapsto \sum f'(x_i) y_i$$

Then m is a kernel of p , that is,

$$\mathcal{E} = (0 \rightarrow J \otimes_A B \xrightarrow{m} B' \xrightarrow{p} B \rightarrow 0)$$

is an exact sequence of A' -modules. Equivalently, there exists a B -module automorphism $\phi : J \otimes_A B \rightarrow J \otimes_A B$ such that $m = k \circ \phi : J \otimes_A B \rightarrow B'$.

Proof (1) $\Rightarrow$ (2) : Let $k : J \otimes_A B \rightarrow B'$ be a B -linear map which is a kernel of p . We show that there exists a unique B -linear homomorphism $\phi : J \otimes_A B \rightarrow J \otimes_A B$

which makes the following diagram commute.

$$\begin{array}{ccccccc}
 J \otimes_A B & \xrightarrow{m} & B' & \xrightarrow{p} & A \otimes_{q, A', f'} B' & \rightarrow & 0 \\
 \downarrow \phi & & \parallel & & \downarrow (f, p) & & \\
 0 \rightarrow J \otimes_A B & \xrightarrow{k} & B' & \xrightarrow{p} & B & \rightarrow & 0
 \end{array}$$

The top row is right-exact as it is obtained by applying $- \otimes_{A'} B$ to $0 \rightarrow J \rightarrow A' \rightarrow A \rightarrow 0$. The bottom row is exact by assumption. The right vertical arrow is an isomorphism by (1). By the universal property of kernel, a unique $\phi : J \otimes_A B \rightarrow J \otimes_A B$ exists that makes the diagram commute. By the right-exactness of the first row, the map $m : J \otimes_A B \rightarrow B'$ surjects onto the kernel of p . Hence the induced map ϕ is surjective. As A is noetherian and $J \otimes_A B$ is a finite A -module, it follows that its surjective endomorphism ϕ is an isomorphism. Hence (2) is proved.

(2) $\Rightarrow$ (1) : The image of $m : J \otimes_A B \rightarrow B'$ is the ideal $f'(J)B' \subset B'$. Hence $A \otimes_{A'} B = (A'/J) \otimes_{A'} B = B'/f'(J)B' = B$ as desired. $\square$

3.18 CAUTION! Suppose that the underlying module of $\ker(p)$ is isomorphic to the module $J \otimes_A B$. This much information **does not by itself** define an element of $Exal_{A'}(B, J \otimes_A B)$. Rather, in order to get an element of $Exal_{A'}(B, J \otimes_A B)$, we have to *choose* a B' -linear map $k : J \otimes_A B \rightarrow B'$ which is a kernel of $p : B' \rightarrow B$. For example, if we change k to $-k$, then $-k : J \otimes_A B \rightarrow B'$ is another kernel of p , and the corresponding element of $Exal_{A'}(B, J \otimes_A B)$ changes sign.

However, if the underlying module of $\ker(p)$ is isomorphic to the module $J \otimes_A B$, then by statement 3.17, the map $m : J \otimes_A B \rightarrow B'$ is a kernel of p , and this canonical choice of the kernel gives us a particular element $[\mathcal{E}] \in Exal_{A'}(B, J \otimes_A B)$.

3.19 Let A' be a ring, $j : J \hookrightarrow A'$ an ideal with $J^2 = 0$, $q : A' \rightarrow A'/J = A$ the quotient, and let $f : A \rightarrow B$ be a ring homomorphism. Let B be regarded as an A' -algebra via the composite $h = f \circ q : A' \rightarrow B$. Let $f' : A' \rightarrow B'$ be an A' -algebra, and let $p : B' \rightarrow B$ be a surjective A' -algebra homomorphism such that $\ker(p)^2 = 0$, and moreover let there be given a B' -linear map $k : J \otimes_A B \rightarrow B'$ which makes

$$\mathbf{E} = (0 \rightarrow J \otimes_A B \xrightarrow{k} B' \xrightarrow{p} B \rightarrow 0)$$

an exact sequence of B' -modules.

The algebra extension $\mathbf{e} = (0 \rightarrow J \xrightarrow{j} A' \xrightarrow{q} A \rightarrow 0)$, where $j : J \rightarrow A'$ denotes the inclusion, defines an element $[\mathbf{e}] \in Exal_{A'}(A, J)$. Let $u : J \rightarrow J \otimes_A B$ be the A' -module map defined by $x \mapsto x \otimes 1$. Consider the square-zero extension

$$u_* \mathbf{e} = (0 \rightarrow J \otimes_A B \xrightarrow{(0, u)} \frac{A' \oplus (J \otimes_A B)}{(j, -u)J} \xrightarrow{(q, 0)} A \rightarrow 0)$$

which is the push-out of $\mathbf{e}$ under u , with $u_*[\mathbf{e}] \in Exal_{A'}(A, J \otimes_A B)$ its extension class.

Then the following two statements are equivalent.

- (a) Under the homomorphism $f^* : \text{Exal}_{A'}(B, J \otimes_A B) \rightarrow \text{Exal}_{A'}(A, J \otimes_A B)$, we have $f^*[\mathbf{E}] = u_*[\mathbf{e}]$.
- (b) The map $k : J \otimes_A B \rightarrow B'$ is identical to the map $m : J \otimes_A B \rightarrow B' : x \otimes p(y) \mapsto f'(x)y$.

Proof By definition of the maps, (a) holds if and only if there exists an A' -algebra isomorphism ψ which makes the following diagram commute.

$$\begin{array}{ccccccc} 0 \rightarrow & J \otimes_A B & \xrightarrow{i_2} & \frac{A' \oplus (J \otimes_A B)}{(j, -u)J} & \xrightarrow{p} & B & \rightarrow 0 \\ & \parallel & & \downarrow \psi & & \parallel & \\ 0 \rightarrow & J \otimes_A B & \xrightarrow{(0, k)} & A \times_{f, B, p} B' & \xrightarrow{\pi_1} & B & \rightarrow 0 \end{array}$$

Note that the first row in the above diagram represents $u_*\mathbf{e}$, while the second row represents $f^*\mathbf{E}$. As ψ is an A' -algebra homomorphism, for all $a' \in A'$ we must have $\psi(a', 0) = (q(a'), f'(a'))$. As the left-hand square commutes, for all $x \in J$ and $y \in B'$ we must have $\psi(0, x \otimes p(y)) = (0, k(x \otimes p(y)))$. Hence ψ must be given by the formula

$$(a', x \otimes p(y)) \mapsto (q(a'), f'(a') + k(x \otimes p(y))).$$

For ψ to be well-defined on the quotient $(A' \oplus (J \otimes_A B))/(j, -u)J$, it must vanish on elements of the form $(x, -x \otimes 1)$. Hence we must have

$$f'(x) - k(x \otimes 1) = 0$$

As k is B -linear, for any $y \in B'$ we must have

$$k(x \otimes p(y)) = k(x \otimes 1)y = f'(x)y$$

Hence the map $k : J \otimes_A B \rightarrow B'$ is identical to the map $m : J \otimes_A B \rightarrow B'$. Conversely, if $k = m$, we can define ψ by the formula

$$\psi(a', x \otimes p(y)) \mapsto (q(a'), f'(a') + f'(x)y)$$

By five lemma, ψ is an isomorphism, proving (b). $\square$

Remark. This remark is included because some authors use $\text{id}_{J \otimes_A B}$ in place of our $u_*[\mathbf{e}]$ so we explain why this is equivalent. With notation as in the above statement, consider the composite isomorphism

$$\text{Exal}_{A'}(A, J \otimes_A B) \rightarrow \text{Hom}_A(H^{-1}(L_{A/A'}), J \otimes_A B) = \text{Hom}_A(J, J \otimes_A B) \rightarrow \text{Hom}_B(J \otimes_A B, J \otimes_A B)$$

where the first map is an isomorphism as $H^0(L_{A/A'}) = \Omega_{A/A'} = 0$ as follows from the small term exact sequence for the spectral sequence for $\mathbb{H}^1$, the middle equality follows from $H^{-1}(L_{A/A'}) = J$, and the third map is the adjunction isomorphism. The image of $u_*[\mathbf{e}]$ in $\text{Hom}_A(J, J \otimes_A B)$ is the map u , and the image of u in $\text{Hom}_B(J \otimes_A B, J \otimes_A B)$ equals $\text{id}_{J \otimes_A B}$. Hence under the composite map, $u_*[\mathbf{e}] \mapsto \text{id}_{J \otimes_A B}$.

3.20 Local criterion for flatness. Let $J \subset A'$ be a nilpotent ideal and let M be an A' -module. The easy case of the local criterion for flatness implies that the following two statements are equivalent.

- (1) M is flat over R .
- (2) M/JM is flat over $A = A'/J$ and the map $J \otimes_{A'} M \rightarrow M : x \otimes m \mapsto xm$ is injective (equivalently, $\text{Tor}_1^{A'}(A'/J, M) = 0$).

3.21 Let A' be a ring, $j : J \hookrightarrow A'$ a finite ideal with $J^2 = 0$, $q : A' \rightarrow A'/J = A$ the quotient, and let $f : A \rightarrow B$ be a ring homomorphism where B is noetherian. Then we have the following.

(1) Let $f' : A' \rightarrow B'$ be an A' -algebra and let $p : B' \rightarrow B$ be an A' -algebra homomorphism which induces an isomorphism $(f, p) : A \otimes_{A'} B' \rightarrow B$ (that is, $p : B' \rightarrow B$ is surjective with kernel $f'(J)B' \subset B'$). If B' is flat over A' , then $m : J \otimes_A B \rightarrow B' : x \otimes p(y) \mapsto f'(x)y$ is injective, so m is a kernel of p . This follows by applying $- \otimes_{A'} B'$ to $\mathbf{e} = (0 \rightarrow J \xrightarrow{j} A' \xrightarrow{q} A \rightarrow 0)$.

(2) Let $f' : A' \rightarrow B'$ be an A' -algebra, and let $\mathbf{E} = (0 \rightarrow J \otimes_A B \xrightarrow{k} B' \xrightarrow{p} B \rightarrow 0)$ be a square-zero extension of A' -algebras, with extension class $[\mathbf{E}] \in \text{Exal}_{A'}(B, J \otimes_A B)$. Suppose that $[\mathbf{E}] \mapsto u_*[\mathbf{e}]$ under the map $f^* : \text{Exal}_{A'}(B, J \otimes_A B) \rightarrow \text{Exal}_{A'}(A, J \otimes_A B)$, where $\mathbf{e}$ is as above. Then by statement 3.19, the map k is identical to the map m , and hence by statement 3.17, the induced map $(f, p) : A \otimes_{A'} B' \rightarrow B$ is an isomorphism. If moreover B is flat over A , then by the above injectivity of m and by the local criterion of flatness (statement 3.20), it follows that B' is flat over A' .

3.22 Lifting theorem for an algebra. Let A' be a commutative ring, $j : J \hookrightarrow A'$ a finitely generated ideal with $J^2 = 0$, $q : A' \rightarrow A'/J = A$ the quotient. Let $f : A \rightarrow B$ be a ring homomorphism where B is noetherian.

Let $\partial : \text{Ext}^1(L_{A/A'}, J \otimes_A B) \rightarrow \text{Ext}^2(L_{B/A}, J \otimes_A B)$ be the connecting homomorphism induced by the distinguished triangle $L_{A/A'} \otimes_A B \rightarrow L_{B/A'} \rightarrow L_{B/A} \xrightarrow{(1)}$.

(I) Existence of a lift of an algebra. Let $\text{obs}_{f,q} \in \text{Ext}^1(L_{B/A}, J \otimes B)$ denote the element

$$\text{obs}_{f,q} = u_*[\mathbf{e}]$$

where $[\mathbf{e}] \in \text{Ext}^1(L_{A/A'}, J)$ denote the class of $0 \rightarrow J \xrightarrow{j} A' \xrightarrow{q} A \rightarrow 0$, and $u : J \rightarrow J \otimes_A B$ denotes the map $x \mapsto x \otimes 1$. Then a lift (B', f', p) of $f : A \rightarrow B$ along $q : A' \rightarrow A$ that satisfies (1) and (2) of statement 3.13 exists if and only if

$$\partial(\text{obs}_{f,q}) = 0 \in \text{Ext}^2(L_{B/A}, J \otimes_A B).$$

Moreover, if $f : A \rightarrow B$ is flat, then any lift $f' : A' \rightarrow B'$ that satisfies (1) and (2) of statement 3.13 is automatically flat.

(II) Set of all lifts. The set of all such lifts (f', p) is a principal set under the canonical action of the group $\text{Ext}^1(L_{B/A}, J \otimes_A B) = \text{Exal}_A(B, J \otimes_A B)$.

(III) Group of all automorphisms. *The group of all automorphisms of any lift (f', p) is canonically isomorphic to $\text{Ext}^0(L_{B/A}, J \otimes_A B) = \text{Der}_A(B, J \otimes_A B)$.*

Proof (I) Existence of a lift: We have an exact triangle $L_{A/A'}^L \otimes_A B \rightarrow L_{B/A'} \rightarrow L_{B/A} \xrightarrow{(1)}$. Applying $\text{Hom}_B(-, J \otimes_A B)$ we get an exact sequence

$$\text{Ext}^1(L_{B/A'}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{A/A'}^L \otimes_A B, J \otimes_A B) \xrightarrow{\partial} \text{Ext}^2(L_{B/A}, J \otimes_A B)$$

We have shown that a lift (B', f', p) is the same as an element of $\text{Exal}_{A'}(B, J \otimes_A B)$ which maps to $u_*[\mathbf{e}]$ under the map $f^* : \text{Exal}_{A'}(B, J \otimes_A B) \rightarrow \text{Exal}_{A'}(A, J \otimes_A B)$. Hence by exactness of $\text{Ext}^1(L_{B/A'}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{A/A'} \otimes_A B, J \otimes_A B) \xrightarrow{\partial} \text{Ext}^2(L_{B/A}, J \otimes_A B)$, a lift exists if and only if

$$\partial(u_*[\mathbf{e}]) = 0 \in \text{Ext}^2(L_{B/A}, J \otimes_A B).$$

(II) Set of all lifts: The set of all lifts is the inverse image of $u_*[\mathbf{e}]$ under the map $\text{Ext}^1(L_{B/A'}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{A/A'} \otimes_A B, J \otimes_A B)$. As the sequence $\text{Ext}^0(L_{A/A'} \otimes_A B, J \otimes_A B) \rightarrow \text{Ext}^1(L_{B/A}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{B/A'}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{A/A'} \otimes_A B, J \otimes_A B)$ is exact, and as $\text{Ext}^0(L_{A/A'} \otimes_A B, J \otimes_A B) = \text{Der}_{A'}(A, J \otimes_A B) = 0$ as $A' \rightarrow A$ is surjective, the sequence $0 \rightarrow \text{Ext}^1(L_{B/A}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{B/A'}, J \otimes_A B) \rightarrow \text{Ext}^1(L_{A/A'} \otimes_A B, J \otimes_A B)$ is exact. Hence the inverse image of $u_*[\mathbf{e}]$ in $\text{Ext}^1(L_{B/A'}, J \otimes_A B)$ is a coset of the subgroup $\text{Ext}^1(L_{B/A}, J \otimes_A B)$, and hence it is a principal set under the group $\text{Ext}^1(L_{B/A}, J \otimes_A B) = \text{Exal}_A(B, J \otimes_A B)$.

(III) Group of all automorphisms: We have already shown in statement 3.16 that all automorphisms of a lift (B', f', p) form the group $\text{Der}_A(B, J \otimes_A B)$. $\square$

3.23 Lifting theorem for a ring homomorphism. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. Let M be a C -module, and let $0 \rightarrow M \rightarrow C' \rightarrow C \rightarrow 0$ be a square-zero extension of A -algebras. Suppose that the following square is commutative.*

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f \uparrow & & \uparrow \\ A & \rightarrow & C' \end{array}$$

Then we can functorially associate an element (defined below in the course of the proof, and called ‘obstruction class for lifting g ’)

$$\text{obs}_{g, C' \rightarrow C} \in \text{Exal}_A(B, g_* M)$$

which has the property that g admits a lift $g' : B \rightarrow C'$ over A (where a ‘lift’ means that the south-east arrow $g' : B \searrow C'$, when added to the above commutative square, should give a new commutative diagram) if and only if $\text{obs}_{g, C' \rightarrow C} = 0$.

Moreover, the set of all lifts is a principal set under $\text{Der}_A(B, M)$.

Proof We put $\text{obs}_{g, C' \rightarrow C} = g^*(\mathbf{e})$ where $\mathbf{e} = (0 \rightarrow M \rightarrow C' \rightarrow C \rightarrow 0) \in \text{Exal}_A(C, M)$, and $g^* : \text{Exal}_A(C, M) \rightarrow \text{Exal}_A(B, g_* M)$ is the pullback under g . The statement about the set of all lifts is clear. $\square$