

1 Derivations and Differentials

Lecture 1 was given on 18/09/2026. The following are notes for lecture 2, to be given on 21/09/2026. Corrections/suggestions are welcome.

2 Cotangent complexes and local properties of morphisms

A first look at cotangent complexes

2.1 What is missing so far? The examples in the previous section show that the first and second Kähler sequences (see ?? and ??) are sometimes exact and sometimes not so. The same is the case for both of the derivation sequences (see ??). What is missing, at the very least, is a universal functorial characterization of the kernels (or cokernels). Such a characterization is provided by cotangent complexes. In fact, it will turn out that cotangent complexes actually do a lot more.

2.2 What is a cotangent complex? To each ring homomorphism $A \rightarrow B$, there is functorially associated an object $L_{B/A}$ of the subcategory $D^{\leq 0}(B) \subset D^-(B)$ of the derived category $D^-(B)$ of bounded-above complexes of B -modules. This is cochain complex

$$L_{B/A} = (\dots \rightarrow L_{B/A}^{-2} \rightarrow L_{B/A}^{-1} \rightarrow L_{B/A}^0 \rightarrow 0)$$

of **flat B -modules**, together with a **functorial homomorphism** $L_{B/A} \rightarrow \Omega_{B/A}$ where the module $\Omega_{B/A}$ is regarded as the complex with the term $\Omega_{B/A}$ in degree 0, and all other terms 0. This induces a functorial isomorphism

$$H^0(L_{B/A}) = L_{B/A}^0 / \text{im}(L_{B/A}^{-1}) \xrightarrow{\sim} \Omega_{B/A}.$$

The functorial properties of the association $(A \rightarrow B) \mapsto (L_{B/A}, L_{B/A} \rightarrow \Omega_{B/A})$ come from its definition. Briefly, $L_{B/A}$ is the value on the object $(A \rightarrow B \xrightarrow{\text{id}} B)$ of the **total left derived functor** – in the homotopical algebra sense – of the **abelianization functor** which is the left part of a certain pair of adjoint functors that form a **Quillen pair**. The properties of $L_{B/A}$ ultimately come from the Quillen pair. In particular, the universal derivation is encoded in the unit of adjunction η in the Quillen pair.

Therefore to each ring homomorphism $A \rightarrow B$ and B -module M , there is functorially associated a sequence of B -modules $D_n(B/A, M) = H^{-n}(L_{B/A} \otimes_B M)$ called the **André-Quillen homology modules** of B over A with coefficients M . Also, there is functorially associated a sequence of B -modules $D^n(B/A, M) = \text{Hom}_{D^{\leq 0}(B)}(L_{B/A}, M[-n])$, called the **André-Quillen cohomology modules** of B over A with coefficients M . Note that $D_n(B/A, M) = 0 = D^n(B/A, M)$ for all $n \leq -1$. Moreover, we have functorial isomorphisms $D_0(B/A, M) = \Omega_{B/A} \otimes_B M$ and $D^0(B/A, M) = \text{Der}_A(B, M)$.

Some history of the developments

2.3 Cartier's theorem (1955) on separable field extensions in modern re-formulation in terms of D_1 : Let $k \subset K$ be an extension of fields. Then

$$tr.deg(L/K) = \dim_L(\Omega_{L/K}) - \dim_L(D_1(L/K, L)).$$

Moreover, the following two statements are equivalent.

- (1) L/K is **separable**, that is, every finitely generated subextension of L admits a **separating transcendental basis** over K .
- (2) $D_1(L/K, L) = 0$ and $tr.deg(L/K) = \dim_L(\Omega_{L/K})$.

2.4 Work of U. Shukla, 1961. It is mentioned in the 1966 paper [L-S] of Lichtenbaum and Schlessinger that these (co-)homology modules D_n and D^n were first invented by U. Shukla in 1961 for the more general context of associative algebras, in the paper [US].

2.5 Work of Grothendieck, 1961. Grothendieck invented the idea of cotangent complexes, and ahead of having the full complex, he directly constructed a 2-term complex $\mathcal{L}_{X/S}^{Grot}$ in special cases (including the affine case), which turns out to be the truncation $\tau_{\geq -1}L_{X/S}$. He applied it to characterizing smooth morphisms, to extending the Grothendieck-Riemann-Roch theorem and relative Serre duality to l.c.i. morphisms between possibly singular varieties, and also to infinitesimal lifting problems.

2.6 Some idea about application to Grothendieck-Riemann-Roch. Explaining this will take us too far, and parts of it are beyond my own expertise, so here is just a very rough sketch of the ideas. I assume the reader will understand the simplified notation that I use here (for example, $A(X)$ is the cohomology or intersection theory with $\mathbb{Q}$ -coefficients, etc). If X, Y are smooth varieties over a field k and $f : X \rightarrow Y$ a proper morphism over k , then we have a (non-commutative) square

$$\begin{array}{ccc} K(X) & \xrightarrow{ch_X} & A(X) \\ f_K \downarrow & & \downarrow f_A \\ K(Y) & \xrightarrow{ch_Y} & A(Y) \end{array}$$

For any coherent sheaf E on X the GRR formula says that

$$ch_Y(f_K(E)) \cdot td(T_Y) = f_A(ch_X(E) \cdot td(T_X))$$

If moreover f is smooth then via multiplicativity of the Todd class and the projection formula, the above becomes

$$ch_Y(f_K(E)) = f_A(ch_X(E) \cdot td(T_X) \cdot f^A(td(T_Y))^{-1}) = f_A(ch_X(E) \cdot td(T_f))$$

where $T_f = \Omega_f^\vee$ is the relative (vertical) tangent bundle on X . If instead of being smooth, f is l.c.i. then the cotangent complex $L_f = L_{X/Y}$ is perfect of amplitude $[-1, 0]$. Writing it as $0 \rightarrow L_f^{-1} \rightarrow L_f^0 \rightarrow 0$, where the L_f^i are chosen to be finite rank locally free $\mathcal{O}_X$ -modules, we have $[\Omega_f] = [L_f^0] - [L_f^{-1}]$ in $K(X)$. Hence a ‘virtual’ relative tangent bundle T_f can be defined using the dual of the cotangent complex $L_{X/Y}$, with $[T_f]$ defined as the element $[(L_f^0)^\vee] - [(L_f^{-1})^\vee]$ in $K(X)$. Its Todd class $td(T_f) = td[(L_f^0)^\vee] \cdot (td[(L_f^{-1})^\vee])^{-1}$ can then be used to re-write the original formula $ch_Y(f_K(E)) \cdot td(T_Y) = f_A(ch_X(E) \cdot td(T_X))$ in the form $ch_Y(f_K(E)) = f_A(ch_X(E) \cdot td(T_f))$. If X and Y are not smooth over k but only l.c.i. over k , then we can similarly use the perfect complexes $L_{X/k}$ and $L_{Y/k}$ to define virtual tangent bundles T_X^{vir} and T_Y^{vir} in $K(X)$ and define their Todd classes. Now provided some more assumptions hold (for example, Rf_*E will have to be a bounded perfect complex in $D_{qc}(Y)$ for its Chern character to make sense), this can be used to generalize G-R-R to l.c.i. morphisms between non-smooth varieties (where the generalization needs to be suitably interpreted in the bivariant intersection theory of Baum-Fulton-MacPherson). I will say no more, given my limited understanding of the issues involved here.

2.7 Work of Lichtenbaum and Schlessinger, 1967. By an imaginative ad hoc construction, Lichtenbaum and Schlessinger directly constructed the truncation $\tau_{\geq -2}L_{B/A}$. They identified the (co-)homological obstruction to a homomorphism $A \rightarrow B$ being an l.c.i. morphism in terms of D_2 and D^2 . However, everything was restricted to the affine case.

2.8 The 1960s. In the present section, we will discuss the truncated cotangent complexes $\mathcal{L}_{B/A}^{Grot} = \tau_{\geq -1}L_{B/A}$ and $\mathcal{L}_{B/A}^{LiSc} = \tau_{\geq -2}L_{B/A}$ and their applications characterization of smooth and to l.c.i. morphisms. The truncated complex $\mathcal{L}_{B/A}^{Grot}$ was constructed by Grothendieck in 1961, with the idea of eventually constructing a full cotangent complex. But that required new ideas, not available in 1961, such as **derived categories** which were soon defined by his student Verdier in his thesis. In the affine case, the full cotangent complex $L_{B/A}$, as an object of $D^{\leq 0}(B)$, was constructed via simplicial methods, independently by M. André and D. Quillen in mid 1960s. For this, André used the so called **co-triple resolutions**, while Quillen invented a conceptual approach via a systematic development of an abstract homotopy theory together with a ‘non-abelian’ version of homological algebra, which he named as **homotopical algebra**. Simplicial methods nicely fit in this general framework. The category $A\text{-alg}/B$ associated to a ring homomorphism $A \rightarrow B$ is not an abelian category. On that we have the **abelianization functor** $A\text{-alg}/B \rightarrow B\text{-mod}$ that sends $(A \rightarrow R \rightarrow B) \mapsto B \otimes_R \Omega_{R/A}$. This is left-adjoint to the functor $B^+ : B\text{-mod} \rightarrow A\text{-alg}/B$ that sends M to an object $B \ltimes M$ of $A\text{-alg}/B$, which has the underlying additive group $B \oplus M$ with multiplication defined by taking $M^2 = 0$, and comes with natural homomorphisms $A \rightarrow B \ltimes M \rightarrow B$ that make it an object of $A\text{-alg}/B$. Quillen’s theory starts by embedding the categories $A\text{-alg}/B$ and $B\text{-mod}$ into the corresponding categories of simplicial objects $sA\text{-alg}/B$ and

$sB\text{-mod}$. By the Dold-Kan theorem, the latter is equivalent to $Ch_{\geq 0}(B)$. The above adjoint functors then define an adjoint pair $sA\text{-alg}/B \rightleftarrows sB\text{-mod}$ with nice properties (that make it what is called a **Quillen pair**), which enable the construction of the **total left derived functor** of the abelianization functor, which is a certain **right Kan extension**.

Note. Left derived functors are **right Kan extensions**, while **right derived functors** are **left Kan extensions**. This reversal is naturally memorable!

This takes the value $L_{B/A} \in \text{Ob } D^{\leq 0}(B)$ on the object $B = (A \rightarrow B \xrightarrow{\text{id}} B)$ of $A\text{-alg}/B$. The natural transformation that is part of the right Kan extension gives the adjunction $L_{B/A} \rightarrow \Omega_{B/A}$. The Quillen adjunction between the total left derived functor of sAb and the total right derived functor of sB^+ contains information about universal derivations.

2.9 Illusie's thesis (1971) uses homotopical theory to develop a **global version** $L_{X/S}$ of cotangent complexes for geometric morphisms $X \rightarrow S$ of **ringed topoi** including schemes, with applications to various deformation problems. Illusie's work brought the developments of the 1960s to a grand conclusion. We will look at it in sections 5 and 6.

2.10 Remark on Globalization. The usual Zariski descent **does not hold** in general for derived categories of quasi-coherent sheaves on Noetherian schemes. So to make $L_{X/S}$ for a scheme X/S we either need a sophisticated argument based on simplicial theory to 'glue' together the affine constructs over affine hypercoverings to have effective cohomological descent in the (cohomologically) quasicohherent bounded above cartesian derived category of $\mathcal{O}$ -modules on the hypercover, or alternatively following Illusie, construct relative cotangent complexes for morphisms of ringed topoi. The simplicial method would have occupied too much space in these elementary notes, so we restrict ourselves to presenting a brief outline of Illusie's method (which also needs simplicial inputs).

A look ahead. In Sections 4 and 5, we sketch some basic concepts of homotopical algebra and we outline the construction of cotangent complexes, and the proofs of their basic properties. The sections 3 and 6 are devoted to some basic applications of cotangent complexes.

How $L_{B/A}$ behaves under a change of $A \rightarrow B$

2.11 Change of rings. The cotangent complexes enjoy nice properties w.r.t. change of rings. To any commutative square of ring homomorphisms

$$\begin{array}{ccc} B & \rightarrow & D \\ \uparrow & & \uparrow \\ A & \rightarrow & C \end{array}$$

there is functorially associated a morphism $D \overset{L}{\otimes}_B L_{B/A} \rightarrow L_{D/C}$ in $D^{\leq 0}(D)$ (the derived tensor product $- \overset{L}{\otimes}_B L_{B/A}$ in this case is just the ordinary tensor product $-\otimes_B L_{B/A}$ because each $L_{B/A}^n$ is a flat B -module), that commutes with the adjunctions $L_{B/A} \rightarrow \Omega_{B/A}$ and $L_{D/C} \rightarrow \Omega_{C/D}$, and the natural map $D \overset{L}{\otimes}_B \Omega_{B/A} \rightarrow \Omega_{D/C}$. This has the obvious composition property w.r.t. the **juxtaposition of two commutative squares** (see the commutative diagram under the heading ‘Exact triangles’ below which has six rings).

Base change under tor-independence. In the above square, let $D = B \otimes_A C$ together with the natural homomorphisms $B \rightarrow B \otimes_A C$ and $C \rightarrow B \otimes_A C$. Suppose that $\text{Tor}_i^A(B, C) = 0$ for all $i \geq 1$ (means B and C are **tor-independent over A**). Then the associated morphism $D \otimes_B L_{B/A} \rightarrow L_{D/C}$ in $D^{\leq 0}(D)$ is an isomorphism.

Note on base change without tor-independence. In the general case where B and C are not necessarily tor-independent over A , one must go over to the category $sA\text{-alg}$ of simplicial A -algebras in which a derived tensor product $B \overset{L}{\otimes}_A C$ exists, and the theory of cotangent complexes appropriately generalizes, giving a base-change formula in general. We will not address it here now.

Exact triangles. To any diagram of ring homomorphisms $(A \rightarrow B \rightarrow C)$, there is functorially associated a connecting homomorphism $L_{C/B} \rightarrow L_{B/A} \otimes_B C[1]$ in $D^-(C)$ (in fact, in $D^{\leq 0}(B)$, as the left translate $L_{B/A}[n]$ is again in $D^{\leq 0}(B)$ for $n \geq 0$). The connecting homomorphism gives an exact triangle $L_{B/A} \otimes_B C \rightarrow L_{C/A} \rightarrow L_{C/B} \rightarrow L_{B/A} \otimes_B C[1]$ in the triangulated category $D^-(C)$ with good functorial properties w.r.t. a morphism $(A \rightarrow B \rightarrow C) \rightarrow (A' \rightarrow B' \rightarrow C')$ of diagrams of ring homomorphisms. This means to any commutative diagram of ring homomorphisms

$$\begin{array}{ccccc} A' & \rightarrow & B' & \rightarrow & C' \\ \uparrow & & \uparrow & & \uparrow \\ A & \rightarrow & B & \rightarrow & C \end{array}$$

there is functorially associated a morphisms of the corresponding exact triangles $(L_{B'/A'} \overset{L}{\otimes}_{B'} C' \rightarrow L_{C'/A'} \rightarrow L_{C'/A'} \xrightarrow{(1)} \rightarrow (L_{B/A} \overset{L}{\otimes}_B C \rightarrow L_{C/A} \rightarrow L_{C/A} \xrightarrow{(1)})$.

2.12 Connecting homomorphisms and long exact sequences. Applying the cohomological functors $\text{Hom}(-, N)$ and $H^0(- \otimes_B N)$ to the above exact triangle in $D^-(C)$ gives C -module homomorphisms $\delta : D_{n+1}(C/B, N) \rightarrow D_n(B/A, N)$ and $\delta : D^n(C/B, N) \rightarrow D^{n+1}(B/A, N)$ for each n , which are called the **connecting homomorphisms**. Moreover, the following two sequences are exact.

$$D_{n+1}(C/B, N) \rightarrow D_n(B/A, N) \rightarrow D_n(C/A, N) \rightarrow D_n(C/B, N) \rightarrow D_{n-1}(B/A, N)$$

$$D^{n-1}(B/A, N) \rightarrow D^n(C/B, N) \rightarrow D^n(C/A, N) \rightarrow D^n(B/A, N) \rightarrow D^{n+1}(C/B, N)$$

Given a morphism of sequences $(A \rightarrow B \rightarrow C) \rightarrow (A' \rightarrow B' \rightarrow C')$ as above and a C -homomorphism $N \rightarrow N'$ from a C module N to a C' -module N' , the induced morphism of exact triangles gives a functorial chain map between the corresponding long exact sequences for the (D_n) s and for the (D^n) s.

Hence, the cotangent complexes answer our above question 2.1 about exact functorial prolongation of the Kähler sequences or the derivation (dual Kähler) sequences.

The extension-lift problem

The extension-lift problem repeatedly occurs in many places in mathematics. The abstract formulation of the problem in categorical terms is as follows. Suppose $\mathcal{C}$ is any category. Let there be given any commutative square in $\mathcal{C}$ as in the diagram (a) below.

$$\begin{array}{ccc} & A & \rightarrow X \\ \text{(a)} & \downarrow & \downarrow \\ & B & \rightarrow Y \end{array} \qquad \begin{array}{ccc} & A & \rightarrow X \\ \text{(b)} & \downarrow & \nearrow \searrow \downarrow \\ & B & \rightarrow Y \end{array}$$

A **filler** or a **lift** for the above commutative square (a) is an arrow $B \rightarrow X$ which makes the resulting larger diagram (b) above commutative, that is, both the triangles in the diagram (b) are commutative. Note that if A is taken to be an initial object of $\mathcal{C}$, then the problem of finding a filler reduces to the usual lifting problem. On the other hand, if Y is taken to be a terminal object in $\mathcal{C}$ then the problem of finding a filler reduces to the usual extension problem. The extension-lift problem provides a common framework.

We can consider the **co-filler** problem in $\mathcal{C}$ obtained by reversing all arrows. That means, a filler problem in a category is the same as a co-filler problem in its opposite category.

2.13 Some examples: (1) **Valuative criteria for separatedness and properness** in schemes takes $A \rightarrow B$ to be the inclusion $\text{Spec } K \hookrightarrow \text{Spec } R$ for a valuation ring R . (2) In topology, the **homotopy lifting property** for a map $X \rightarrow Y$ takes B to $B = A \times I$ and $A \rightarrow B$ to be the inclusion $a \mapsto (a, 0)$. (3) The **homotopy extension property** for the inclusion of a subspace $A \hookrightarrow B$ in a topological space B takes X to be the path space Y^I , and takes the map $X \rightarrow Y$ to be the projection $X^I \rightarrow Y$ that sends a path $\gamma : I \rightarrow Y$ to its starting point $\gamma(0)$.

Formally unramified, étale, smooth morphisms

2.14 Definitions. Let $\pi : Y \rightarrow X$ be morphism of schemes. Suppose C is a commutative ring with an ideal I with $I^2 = 0$. Suppose we are given morphisms $\phi : \text{Spec } C \rightarrow X$ and $\psi : \text{Spec}(C/I) \rightarrow Y$ such that the perimeter square in the following diagram commutes.

$$\begin{array}{ccc} \text{Spec}(C/I) & \xrightarrow{\psi} & Y \\ \downarrow & \nearrow \theta & \downarrow \\ \text{Spec } C & \xrightarrow{\phi} & X \end{array}$$

Consider the following conditions.

- (1) There exists at most one morphism $\theta : \text{Spec } C \rightarrow Y$ which makes the above diagram commute.
- (2) There exist at least one morphism $\theta : \text{Spec } C \rightarrow Y$ which makes the above diagram commute.
- (3) There exists exactly one morphism $\theta : \text{Spec } C \rightarrow Y$ which makes the above diagram commute.

The morphism $\pi : Y \rightarrow X$ is said to be **formally unramified** if (1) holds for all squares as above, **formally smooth** if (2) holds for all squares as above, and **formally étale** if (3) holds for all squares as above.

We are interested in the local theory, so we will consider the affine case, where $X = \text{Spec } A$ and $Y = \text{Spec } B$, and thereby translate the above filler question into a co-filler question. Given the commutative square diagram (I) below, we want all co-fillers $\theta : B \rightarrow C$ that make the diagram (II) below commutative.

$$\begin{array}{ccc}
 \text{(I)} & B & \rightarrow C/I \\
 & \uparrow & \uparrow \\
 & A & \rightarrow C
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{(II)} & B & \rightarrow C/I \\
 & \uparrow & \searrow \theta \quad \uparrow \\
 & A & \rightarrow C
 \end{array}$$

2.15 Set of all co-fillers is naturally a $\text{Der}_A(B, I)$ -torsor. The set S of all co-fillers $\theta : B \rightarrow C$ in the diagram (II) above is naturally a principal set under the additive action of

$$\text{Der}_A(B, I) = \text{Hom}_B(\Omega_{B/A}, I)$$

where I is given the B -module structure induced by the homomorphism $B \rightarrow C/I$. If $\theta \in S$ and $v : B \rightarrow I$ is in $\text{Der}_A(B, I)$, then consider the map $\theta + v : B \rightarrow C : b \mapsto \theta(b) + v(b)$. This is clearly an A -algebra homomorphism, and makes the diagram commute, so lies in S . This defines an additive action $S \times \text{Der}_A(B, I) \rightarrow S$, which can be seen to be transitive and free.

Formally unramified homomorphisms and $\tau_{\geq 0} L_{B/A}$

2.16 Proposition Let A be noetherian, and let $A \rightarrow B$ be a ring homomorphism of finite type. Then the following statements (1), (2), (3) are equivalent. The statement (4) implies (1), (2), (3). In the special case where A is noetherian and B is of finite type over A , all four statements are equivalent.

- (1) B is formally unramified over A .
- (2) $\Omega_{B/A} = 0$.
- (3) The diagonal ideal $I_\Delta \subset B \otimes_A B$ is idempotent.
- (4) The diagonal morphism $\text{Spec } B \rightarrow \text{Spec } B \times_{\text{Spec } A} \text{Spec } B$ is an open and closed embedding.

Proof For any B -module M , we have the ring $C = B \ltimes M$, and $I = M \subset C$. Note that $I^2 = 0$. The resulting square admits a filler θ_0 defined by $\theta_0(b) = (b, 0)$. (1) implies that there is at most one filler, and so θ_0 is the unique filler, so the set of all fillers is the singleton set $\{\theta_0\}$. As this set is a $\text{Der}_A(B, M)$ -torsor, it follows that $\text{Der}_A(B, M) = 0$. As this holds for each M the statement (2) follows. Conversely, if $\Omega_{B/A} = 0$, then $\text{Der}_A(B, I) = \text{Hom}_B(\Omega_{B/A}, I) = 0$ so the set of all fillers S is either empty or a singleton, showing that (1) holds. By its second construction, $\Omega_{B/A} = I_\Delta/I_\Delta^2$. Hence (2) and (3) are equivalent. If (4) holds, then $I_\Delta = I_\Delta^2$, so $\Omega_{B/A} = 0$ showing (4) implies (2).

If A is noetherian and B is of finite type over A , then the ring $B \otimes_A B$ is noetherian. An ideal in a noetherian ring is idempotent if and only if it defines an open and closed subscheme in its spectrum. Hence in this special case, (2) is equivalent to (4).

Note: If the diagonal Δ is open and closed then $\Omega_{B/A} = 0$, but the converse may not always hold. (Exercise: Give an example.) $\square$

The Grothendieck cotangent complex $\mathcal{L}_{B/A}^{\text{Grot}} = \tau_{\geq -1} L_{B/A}$

2.17 The ‘good’ truncation $\tau_{\geq n}$. Let $\mathcal{A}$ be an abelian category, $C^*(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$ where $*$ $\in \{\emptyset, +, -, b\}$ (depending on $*$, these will be the unbounded, bounded below, bounded above, or bounded complexes). and let $D^*(\mathcal{A})$ denote the corresponding derived category. For each $n \in \mathbb{Z}$, recall that we have a **truncation functor** $\tau_{\geq n} : C^*(\mathcal{A}) \rightarrow C^*(\mathcal{A})$ that sends an object $M = (\dots \rightarrow M^r \rightarrow M^{r+1} \rightarrow \dots)$ of $C^*(\mathcal{A})$ to the object

$$\tau_{\geq n} M = (\dots \rightarrow 0 \rightarrow (M^n / \text{im } M^{n-1}) \rightarrow M^{n+1} \rightarrow \dots)$$

where $(\tau_{\geq n} M)^r = 0$ for all $r \leq n-1$, $(\tau_{\geq n} M)^n = M^n / \text{im } M^{n-1}$ and $(\tau_{\geq n} M)^r = M^r$ for $R \geq n+1$, with the co-boundary maps induced by those of M . The functor has the obvious definition on arrows (cochain maps). The functor $\tau_{\geq n} : C^*(\mathcal{A}) \rightarrow C^*(\mathcal{A})$ goes down to the corresponding derived category to define an additive functor $\tau_{\geq n} : D^*(\mathcal{A}) \rightarrow D^*(\mathcal{A})$.

Caution. The functors $\tau_{\geq n}$ (and also $\tau_{\leq n}$) do not commute with translations, and do not preserve exact triangles.

2.18 Direct definition of $\mathcal{L}_{B/A}^{\text{Grot}} = \tau_{\geq -1} L_{B/A}$.

Following Grothendieck’s idea from 1961, it is possible to directly associate a complex $\mathcal{L}_{B/A}^{\text{Grot}} \in D^{\geq -1}(B)$ to a ring homomorphism $A \rightarrow B$, which will eventually turn out to be naturally isomorphic to the truncated piece $\tau_{\geq -1} L_{B/A}$ of the full complex $L_{B/A}$. The Stacksproject calls this directly defined complex as the **naïve cotangent complex**, and we will adopt this terminology. This naïve complex was adequate for certain applications that Grothendieck made of it. We now give its direct construction.

Let $A \rightarrow B$ be a ring homomorphism. Let $P = A[x_b]_{b \in B}$ be the polynomial algebra over A in the set of variables $\{x_b\}$ where b varies over B . Let $P \rightarrow B$ be the A -algebra homomorphism that sends $x_b \mapsto b$. This is clearly surjective, and its kernel is the ideal I generated by the following three kinds of elements: (i) $x_a - a$ for $a \in A$, (ii) $x_{b_1+b_2} - x_{b_1} - x_{b_2}$ for $b_1, b_2 \in B$, and (iii) $x_{b_1 b_2} - x_{b_1} x_{b_2}$ for $b_1, b_2 \in B$. The complex $\mathcal{L}_{B/A}^{Grot}$ can be directly defined as the following cochain complex of B -modules and B -linear homomorphisms

$$0 \rightarrow I/I^2 \xrightarrow{d} \Omega_{P/A} \otimes_P P/I \rightarrow 0$$

In the above, the term I/I^2 is in degree -1 , the term $\Omega_{P/A} \otimes_P P/I$ is in degree 0 , and the map d is induced by the universal derivation $d_{P/A} : P \rightarrow \Omega_{P/A}$. Note that the exactness of the second fundamental sequence $I/I^2 \xrightarrow{d} \Omega_{P/A} \otimes_P P/I \rightarrow \Omega_{B/A} \rightarrow 0$ implies that we have a natural isomorphism $H^0(\mathcal{L}_{B/A}^{Grot}) = \Omega_{B/A}$.

2.19 Caution! The way $L_{B/A}$ is usually manufactured, the individual terms $L_{B/A}^n$ are free, hence flat, as B -modules, so the functor $L_{B/A} \overset{L}{\otimes}_B -$ reduces termwise to $L_{B/A}^n \otimes_B -$. However, the n th term $L_{B/A}^n / \text{im } L_{B/A}^{n-1}$ of the truncated complex $\tau_{\geq n} L_{B/A}$ need not be a free or flat B -module. In particular, the (-1) th term of $\mathcal{L}_{B/A}^{Grot}$ need not be free or flat over B . Hence, we must use the derived tensor product $\mathcal{L}_{B/A}^{Grot} \overset{L}{\otimes}_B -$ when tensoring with $\mathcal{L}_{B/A}^{Grot}$.

2.20 $\mathcal{L}_{B/A}^{Grot}$ in the Stacksproject The Stacksproject contains a concrete, easy to read account of the truncated cotangent complex $\tau_{\leq -1} L_{B/A}$, which they call the **naïve cotangent complex**, and denote by $\mathcal{NL}_{B/A}$ (see Stacksproject tag 00S0). They give a fuller and much more detailed treatment of it than what we have sketched here.

2.21 Proposition. Let $A \rightarrow B$ be a ring homomorphism, and let $A \rightarrow Q \rightarrow B$ be a factorization of $A \rightarrow B$ (means Q is an object of $A\text{-alg}/B$) such that Q is a polynomial algebra over A and $Q \rightarrow B$ is surjective. Let $J = \ker(Q \rightarrow B)$. Let $J/J^2 \rightarrow \Omega_{Q/A} \otimes_Q B \rightarrow \Omega_{B/A} \rightarrow 0$ denote the conormal sequence of $A \rightarrow Q \rightarrow B$, and let $\mathcal{E} = (0 \rightarrow J/J^2 \rightarrow \Omega_{Q/A} \otimes_Q B \rightarrow 0)$ denote the corresponding object of the derived category $D^-(B)$ of cohomologically bounded above cochain complexes of B modules, where the term J/J^2 is in the (-1) th position. Let $\mathcal{E} \rightarrow \Omega_{B/A}$ be the morphism in $D^-(B)$ defined by the conormal sequence where for any B -module M as usual we regard M as an object of $D^-(B)$ concentrated in degree 0 . Then there exists a natural isomorphism $\mathcal{E} \rightarrow \mathcal{L}_{B/A}^{Grot}$ in D^- that commutes with the respective homomorphisms to $\Omega_{B/A}$, and is functorial in $A \rightarrow B$.

Proof (Sketch) Let $A \rightarrow P \rightarrow B$ be as in 2.18. Let T be the disjoint union of the set of variables x_b that define P and the set of variables y_i that define Q as

polynomial rings over A . Let R be the polynomial ring in T over A . Let $A \rightarrow R \rightarrow B$ be the corresponding factorization of $A \rightarrow B$, making R an object of $A\text{-alg}/B$ that lies above both P and Q , with surjection $R \rightarrow P$ and $R \rightarrow Q$. We can choose these surjections so that they send the common variables to themselves and the extra variables to zero. They each have a section $P \rightarrow R$ and $Q \rightarrow R$, sending the variables to themselves. Let $M = \ker(R \rightarrow B)$, and let $\mathcal{F} = (0 \rightarrow M/M^2 \rightarrow \Omega_{R/A} \otimes_R B \rightarrow 0)$ be the corresponding object of $D^-(B)$ with the induced homomorphism $C \rightarrow \Omega_{B/A}$. We now leave it to the reader to establish a natural isomorphism $\mathcal{F} \rightarrow \mathcal{L}^{Grot}$ and $\mathcal{F} \rightarrow \mathcal{E}$ in $D^-(B)$ that preserves the augmentations to $\Omega_{B/A}$, and from that conclude the desired result. $\square$

2.22 Help in computation. The A -algebra P can be ‘too large’, making the terms in $\mathcal{L}_{B/A}^{Grot}$ ‘too large’. The above shows that if B is finitely presented over A , then we can choose an isomorphic version $\mathcal{E}$ of $\mathcal{L}^{Grot}$ such that each $\mathcal{E}^n$ is a finitely presented B -modules. This is helpful in concrete calculations.

2.23 The homology $D_1(B/A) = H^{-1}(\mathcal{L}_{B/A}^{Grot})$. This B -module is called the **intrinsic conormal module of B/A** . It will follow from the equality $\mathcal{L}_{B/A}^{Grot} = \tau_{\geq -1} L_{B/A}$ that $H^{-1}(\mathcal{L}_{B/A}^{Grot}) = H^{-1}(L_{B/A}) = D_1(B/A)$, the first André-Quillen homology. If B is finitely presented over A , then it follows that $D_1(B/A)$ is a finitely presented B -module.

Formally smooth homomorphisms and $\tau_{\geq -1} L_{B/A}$

2.24 Compared to the case of formally unramified maps, it is more difficult to characterize formally smooth/étale maps in terms of their differential behaviour. To begin with, we have the following.

- (1) If $A \rightarrow B$ is formally smooth, then for any ring homomorphism $R \rightarrow A$, the induced B -linear map $\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}$ is a direct summand, giving a split exact sequence $0 \rightarrow \Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R} \rightarrow \Omega_{B/A} \rightarrow 0$. The converse is false.
- (2) If $A \rightarrow B$ is formally étale, then for any ring homomorphism $R \rightarrow A$, the induced B -linear map $\Omega_{A/R} \otimes_A B \rightarrow \Omega_{B/R}$ is an isomorphism. The converse is false.

Proof (1): Let $C = B\langle \Omega_{A/R} \otimes_A B \rangle = B \ltimes (\Omega_{A/R} \otimes_A B)$ and let $I = \Omega_{A/R} \otimes_A B$ which is an ideal in C with $I^2 = 0$, with $C/I = B$. Consider the ring homomorphism $\phi : A \rightarrow C$ defined by $a \mapsto (a, da \otimes 1)$, and the resulting commutative square

$$\begin{array}{ccc} B & \xrightarrow{\text{id}_B} & B \\ \uparrow & \searrow \theta & \uparrow \\ A & \xrightarrow{\phi} & C \end{array}$$

By formal smoothness of B/A , there exists a homomorphism $\theta : B \rightarrow C$ making the whole diagram commute. But such a θ is the same as an element of $\theta' \in \text{Hom}_B(\Omega_{B/R}, \Omega_{A/R} \otimes_A B)$ which lifts the identity map of $\Omega_{A/R} \otimes_A B$. In particular, we get a decomposition $\Omega_{B/R} = N \oplus \Omega_{A/R} \otimes_A B$, where $N = \ker(\theta')$, such that $\theta' = p_2$, the second projection. This proves the first statement in (1).

(2): The first statement follows immediately from (1) and statement 2.16.

The converse statements in (1) and (2) are false as seen by taking $R \rightarrow A \rightarrow B$ to be surjections, in which case $\Omega_{A/R}$, $\Omega_{B/R}$ and $\Omega_{B/A}$ are all zero. We can choose the surjection $A \rightarrow B$ not to be formally smooth. For example, take $R = k[x, y]$, $A = k[x, y]/(y) = k[x]$, $B = k[x]/(x) = k$. $\square$

2.25 Proposition. The conormal (means second Kähler) sequence of a closed subscheme of an affine scheme splits if and only if the closed subscheme is a relative first order neighbourhood retract.

Let $A \rightarrow B \rightarrow C$ be ring homomorphisms, such that $B \rightarrow C$ is surjective with kernel ideal I . Then the following statements are equivalent.

(1) The C -module homomorphism $\delta : I/I^2 \rightarrow \Omega_{B/A} \otimes_B C : b \mapsto db \otimes 1$ is a split monomorphism, that is, there exists a C -module homomorphism $\pi : \Omega_{B/A} \otimes_B C \rightarrow I/I^2$ such that $\pi \circ \delta = \text{id}_{I/I^2}$.

(1') The conormal sequence $0 \rightarrow I/I^2 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow 0$ is a split-exact sequence of C -modules.

(2) The first-order tubular neighbourhood $\text{Spec } B/I^2$ of the A -scheme $\text{Spec } C$ in $\text{Spec } B$ has a retract to $\text{Spec } C$ over $\text{Spec } A$, that is, the A -algebra homomorphism $B/I^2 \rightarrow B/I = C$ admits a section $\sigma : C \rightarrow B/I^2$ over A .

Proof. From the more elementary fact that $I/I^2 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow 0$ is exact, it follows that (1) and (1') are equivalent.

Let $E = B/I^2$ and $J \subset E$ be the ideal I/I^2 , so that $C = E/J$. Then we have the following commutative diagram in the category of C -modules, where the rows are the right-exact conormal sequences for $A \rightarrow B \rightarrow C$ and $A \rightarrow E \rightarrow C$.

$$\begin{array}{ccccccc} I/I^2 & \rightarrow & \Omega_{B/A} \otimes_B C & \rightarrow & \Omega_{C/A} & \rightarrow & 0 \\ \parallel & & \psi \downarrow & & \parallel & & \\ J/J^2 & \rightarrow & \Omega_{E/A} \otimes_E C & \rightarrow & \Omega_{C/A} & \rightarrow & 0 \end{array}$$

The map $\psi : \Omega_{B/A} \otimes_B C \rightarrow \Omega_{E/A} \otimes_E C$ sends $d_{B/A}(b) \otimes_B 1 \mapsto d_{E/A}(b) \otimes_E 1$ and we claim that it is an isomorphism (surjectivity is clear). This follows dually from the observation that if N is a B/I -module, equivalently, a B -module with $IN = 0$, then the natural map $\text{Der}_A(B/I^2, N) \rightarrow \text{Der}_A(B, N)$ is an isomorphism (while it is clearly injective, its surjectivity follows from $IN = 0$). As ψ is an isomorphism, the above commutative diagram shows that the conormal sequence for $A \rightarrow B \rightarrow C = B/I$ does not change if B is replaced by the A -algebra $E = B/I^2$. By this device, we can assume that $I^2 = 0$. We will identify C with B/I , and write the quotient map as $q : B \rightarrow B/I : b \mapsto \bar{b}$.

We will go from a section σ to a projection π , or in the reverse direction, so that we have

$$\pi(db \otimes 1) + \sigma(\bar{b}) = b$$

for all $b \in B$.

(1) $\Rightarrow$ (2): Given a left-inverse $\pi : \Omega_{B/A} \otimes_B C \rightarrow I/I^2$ to $\delta : I/I^2 \rightarrow \Omega_{B/A} \otimes_B C : b \mapsto db \otimes 1$, define $\sigma : B/I \rightarrow B$ by the formula $\sigma(\bar{b}) = b - \pi(db \otimes 1)$. This can be seen to be well defined, and give a section of $q : B \rightarrow B/I$ over A .

(2) $\Rightarrow$ (1): Given a section σ of $q : B \rightarrow B/I$ over A , we define a map $v : B \rightarrow I$ by $v(b) = b - \sigma(\bar{b})$. It can be seen that this map is A -derivation, so it induces a B -module homomorphism $\pi' : \Omega_{B/A} \rightarrow I$. This induces a C -module homomorphism $\pi : \Omega_{B/A} \otimes_B C \rightarrow I$. It can be seen that π is a left-inverse to $\delta : I \rightarrow \Omega_{B/A} \otimes_B C$. $\square$

2.26 If $A \rightarrow B$ is formally smooth, then $\Omega_{B/A}$ is a projective B -module. Converse is false.

Proof. We have an A -algebra quotient $P \rightarrow B$ where P is a polynomial ring over A . Let $I = \ker(P \rightarrow B)$. By formal smoothness of B/A , there exists a homomorphism $\theta : B \rightarrow P/I^2$ as the diagonal map which will make the following diagram commute.

$$\begin{array}{ccc} B & = & P/I \\ \uparrow & \searrow \theta & \uparrow \\ A & \rightarrow & P/I^2 \end{array}$$

Hence by 2.25, the conormal sequence $0 \rightarrow I/I^2 \rightarrow \Omega_{P/A} \otimes_P B \rightarrow \Omega_{B/A} \rightarrow 0$ is a split-exact sequence of B -modules. As $P = A[x_i]$ is a polynomial ring over A , the module $\Omega_{P/A}$ is a free P -module with basis dx_i , and so $\Omega_{P/A} \otimes_P B$ is a free B -module. Being its direct summand, $\Omega_{B/A}$ is a free B -module as desired.

To see that the converse is false, take B/A to be a nontrivial purely inseparable finite extension of fields. $\square$

2.27 Equivalent conditions for defining formal smoothness. If $A \rightarrow B$ is a ring homomorphism then the following are equivalent.

- (1) $A \rightarrow B$ is formally smooth as defined in statement 2.14.
- (2) There exists a formally smooth $A \rightarrow C$ and a surjection of A -algebras $C \rightarrow C/I = B$ such that the induced A -algebra homomorphism $C/I^2 \rightarrow B$ admits a section $B \rightarrow C/I^2$ in the category of A -algebras.
- (2') There exists a formally smooth $A \rightarrow C$ and a surjection of A -algebras $C \rightarrow C/I = B$ such that the B -module homomorphism $d : I/I^2 \rightarrow \Omega_{C/A} \otimes_C B$ is a split injection.
- (2'') The B -module $\Omega_{B/A}$ is projective, and there exists a formally smooth $A \rightarrow C$ and a surjection of A -algebras $C \rightarrow C/I = B$ such that the B -module homomorphism $d : I/I^2 \rightarrow \Omega_{C/A} \otimes_C B$ is an injection.

(3) If $A \rightarrow C$ is formally smooth and if there exists a surjection of A -algebras $C \rightarrow C/I = B$ then the induced A -algebra homomorphism $C/I^2 \rightarrow B$ admits a section $B \rightarrow C/I^2$ in the category of A -algebras.

(3') If $A \rightarrow C$ is formally smooth and if there is a surjection of A -algebras $C \rightarrow C/I = B$ then the B -module homomorphism $d : I/I^2 \rightarrow \Omega_{C/A} \otimes_C B$ is a split injection.

(3'') The B -module $\Omega_{B/A}$ is projective, and if $A \rightarrow C$ is formally smooth and if there is a surjection of A -algebras $C \rightarrow C/I = B$ then the B -module homomorphism $d : I/I^2 \rightarrow \Omega_{C/A} \otimes_C B$ is an injection.

(4) $\text{Hom}_{D^-(B)}(L_{B/A}, M[1]) = \text{Ext}_A(B, M) = 0$ for all B -modules M .

(5) $H^{-1}(L_{B/A}) = 0$ and $H^0(L_{B/A}) = \Omega_{B/A}$ is a projective B -module.

(5') The B -module $\Omega_{B/A}$ is projective, and the adjunction map $\mathcal{L}_{B/A}^{\text{Grot}} \rightarrow \Omega_{B/A}$ is a quasi-isomorphism (here, $\mathcal{L}_{B/A}^{\text{Grot}}$ denotes the naïve cotangent complex $\tau_{\geq -1}L_{B/A}$ of $A \rightarrow B$ defined in statement 2.18).

(5'') The naïve cotangent complex $\mathcal{L}_{B/A}^{\text{Grot}}$ is isomorphic in the derived category $D^-(B)$ of B -modules to a complex defined by a projective B -module in degree 0.

2.28 Exercise (Zariski local nature of formally unramified/smooth/étale homomorphisms.) Let $A \rightarrow B$ be a ring homomorphism. Show the following.

(1) If $A \rightarrow B$ is formally unramified/smooth/étale and $S \subset B$ then $A \rightarrow S^{-1}B$ is formally unramified/smooth/étale respectively.

(2) Let $f_1, \dots, f_n \in B$ such that $(f_1, \dots, f_n) = (1)$, the unit ideal. Then $A \rightarrow B$ is formally unramified/smooth/étale if and only if each $A \rightarrow B_{f_i}$ is formally unramified/smooth/étale respectively.

Local complete intersections and $\tau_{\geq -2}L_{B/A}$

The 2-term truncated complex $\tau_{\geq -1}L_{B/A}$ is not enough for the application of $L_{B/A}$ for deciding whether a morphism is a local complete intersection, where the longer 3-term truncated complex $\tau_{\geq -2}L_{B/A}$ is needed.

2.29 Definition: Local complete intersection morphisms. A closed embedding of noetherian schemes $X \hookrightarrow Y$ is called a regular embedding if Zariski locally X is defined by a regular sequence of functions on Y . Let $f : X \rightarrow Y$ be a finite type morphism of noetherian schemes. We say that f is a local complete intersection (l.c.i.) morphism if Zariski locally over X , f factors as a regular immersion followed by a smooth morphism $U \rightarrow Z \rightarrow Y$ ($U \hookrightarrow Z$ is a regular immersion where U is open in X , and $Z \rightarrow Y$ is smooth).

2.30 Theorem: Characterization of l.c.i. morphisms $X \rightarrow Y$ via $L_{X/Y}$.

Let $f : X \rightarrow Y$ be a locally finite type morphism of noetherian schemes. Then the following are equivalent.

- (1) f is an l.c.i. morphism.
- (2) The cotangent complex $L_f = L_{X/Y}$ is locally of perfect amplitude contained in $[-1, 0]$, that is, Zariski locally it can be represented by as $0 \rightarrow E^{-1} \rightarrow E^0 \rightarrow 0$ where the E^i are coherent locally free $\mathcal{O}_X$ -modules.
- (3) The 3-term truncated cotangent complex $\tau_{\geq -2}L_{X/Y}$ is locally of perfect amplitude contained in $[-1, 0]$.

Note. To see a method for calculation of the cotangent complexes of local complete intersections, see Stacksproject, Section 92.14 : The cotangent complex of a local complete intersection.

2.31 Caution. For a finite type homomorphism $f : A \rightarrow B$ of noetherian rings to be l.c.i., it is **necessary but not sufficient** that the truncated cotangent complex $\mathcal{L}_{B/A}^{Grot} = \tau_{\geq -1}L_{B/A}$ is perfect. For example, let $A = k[\epsilon]$ where k is a field and $\epsilon^2 = 0$, and let B be its residue field k .

Exercise. Determine $\mathcal{L}_{B/A}^{Grot}$ in this case, and show that it is perfect of amplitude contained in $[-1, 0]$. But show that $A \rightarrow B$ is not l.c.i. (note that ϵ is a zero divisor in A).

The above example shows the importance of having an easy access to the longer 3-term truncation $\tau_{\geq -2}L_{X/Y}$. This was achieved by Lichtenbaum and Schlessinger in 1967.

2.32 The work of Lichtenbaum-Schlessinger (1967). The paper of Lichtenbaum-Schlessinger gives a very concrete, computable representation of $\tau_{\geq -2}L_{B/A}$ for a finite type homomorphism $A \rightarrow B$ of noetherian rings. They show that their complex is perfect of amplitude contained in $[-1, 0]$ if and only if $A \rightarrow B$ is l.c.i. It turned out that their complex is the truncation $\tau_{\geq -2}L_{B/A}$ of the full cotangent complex which was only constructed later. Hence the equivalence between conditions (1) and (3) in the Theorem 2.30, together with the Lichtenbaum-Schlessinger construction, becomes a practical method for testing for l.c.i. homomorphisms or for calculating D_i and D^i for $0 \leq i \leq 2$.

2.33 The complex $\mathcal{L}_{B/A}^{LiSc}$ of Lichtenbaum-Schlessinger. Let $A \rightarrow B$ be a given homomorphism. A **LiSc two-term extension** of $A \rightarrow B$ consists of the following.

- (1) A polynomial algebra R over A together with a surjective A -algebra homomorphism $e_0 : R \rightarrow B$,
- (2) R -modules E_1 and E_2 and R -module homomorphisms $E_2 \xrightarrow{e_2} E_1 \xrightarrow{e_1} R$ such that the following conditions are satisfied:
 - (a) For all $x, y \in E_1$, we have $e_1(x)y = e_1(y)x \in E_1$.

(b) The following sequence of abelian groups is exact.

$$0 \rightarrow E_2 \xrightarrow{e_2} E_1 \xrightarrow{e_1} R \xrightarrow{e_0} B \rightarrow 0$$

Note that if $I = \ker(e_0)$, then these conditions imply that $IE_2 = 0$, which makes E_2 naturally a B -module. Hence this gives rise to the following 3-term complex of B -modules

$$0 \rightarrow E_2 \xrightarrow{d_2} B \otimes_R E_1 \xrightarrow{d_1} B \otimes_R \Omega_{R/A} \rightarrow 0$$

in which the indexing is so chosen that term $B \otimes_R \Omega_{R/A}$ is in the (0)th place. The complex $\mathcal{L}_{B/A}^{LiSc}$ will be the 3-term complex as above that is obtained by making a special sort of a LiSc two-term extension of $A \rightarrow B$ that they call a **free extension**.

Such a special LiSc two-term extensions (free extension) of $A \rightarrow B$ can be obtained as follows: First choose a surjection $e_0 : R \rightarrow B$ of A -algebras, where R is a polynomial algebra over A . Let $I = \ker(e_0)$. Then choose an exact sequence

$$0 \rightarrow U \xrightarrow{i} F \xrightarrow{j} I \rightarrow 0$$

of R -modules, where F is a free R -module. Let $\varphi : F \otimes_R F \rightarrow F$ be defined by $\varphi(x \otimes y) = j(x)y - j(y)x$, and let $U_0 = \ker(\varphi)$. Finally, let $E_1 = F/U_0$ and $E_2 = U/U_0$ with the induced maps e_1 and e_2 . This defines the complex $\mathcal{L}_{B/A}^{LiSc}$.

2.34 Cohomological criterion. To find out if a finite type homomorphism $f : A \rightarrow B$ of noetherian rings is l.c.i., we need to know if we can locally choose a representative for $L_{B/A}$ of the form

$$\dots \rightarrow E^{-3} \rightarrow 0 \rightarrow E^{-1} \rightarrow E^0 \rightarrow 0$$

with $E^{-2} = 0$ where the E^i are finite locally free B -modules. Simple linear algebra of finite projective modules and homomorphisms (and Nakayama's lemma) translates this into the cohomological criterion of the Theorem 2.36. We first state the simpler version for closed immersions.

2.35 Local complete intersection subscheme. (Lichtenbaum-Schlessinger 1967, Proposition 3.2.1.) Let A be a noetherian ring, $I \subset A$ an ideal and let $A \rightarrow A/I = B$ be the quotient map. Then the following are equivalent:

- (1) For any prime $\mathfrak{q} \subset B$ with inverse image $\mathfrak{p} \subset A$, the kernel of $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is generated by an $A_{\mathfrak{p}}$ -sequence.
- (2) $D^2(B/A, M) = 0$ for all B -modules M .
- (3) $D_2(B/A, B) = 0$ and I/I^2 is a projective B -module.
- (4) $D_2(B/A, M) = 0$ for all B -modules M .

2.36 Theorem. (Lichtenbaum-Schlessinger 1967, Corollary 3.2.2.) Let A be a noetherian ring, $A \rightarrow B$ a homomorphism of finite type. Then the following are equivalent:

- (1) There exists a finite type polynomial algebra P over A and an ideal $I \subset P$ such that $B = P/I$ over A , and the quotient $P \rightarrow P/I$ satisfies the four equivalent conditions of the above Proposition 2.35 (means $\text{Spec } B$ is embeddable as a local complete intersection closed subscheme in some affine space over $\text{Spec } A$).
- (2) If Q is any finite type polynomial algebra over A with an ideal J such that $Q/J = B$ over A , then the quotient $Q \rightarrow Q/J$ satisfies the four equivalent conditions of the above Proposition 2.35.
- (3) $D^2(B/A, M) = 0$ for all B -modules M .
- (4) $D_2(B/A, B) = 0$ and I/I^2 is a projective B -module for the kernel I of any surjection $P \rightarrow B$ over A where P is a finite type polynomial algebra over A .
- (5) $D_2(B/A, M) = 0$ for all B -modules M .

2.37 Quillen Conjectures. Let $A \rightarrow B$ is a finite type homomorphism of noetherian rings.

- (1) If $L_{B/A}$ is perfect of bounded amplitude contained in $[-n, 0]$ for some $n \geq 0$, then $L_{B/A}$ is perfect of bounded amplitude contained in $[-2, 0]$.
- (2) (**Proved by Avramov** in 1999, see arXiv:math/9909192.) If $L_{B/A}$ is perfect of bounded amplitude and B is of finite flat dimension over A , then $A \rightarrow B$ is l.c.i.

2.38 Summary. Let $A \rightarrow B$ be a finite type homomorphism of noetherian rings. Then the following holds.

- (1) $A \rightarrow B$ is unramified if and only if $\Omega_{B/A} = \tau_{\geq 0} L_{B/A} = 0$.
- (2) $A \rightarrow B$ is smooth if and only if $\mathcal{L}_{B/A}^{Grot} = \tau_{\geq -1} L_{B/A}$ is of perfect amplitude contained in $[0]$.
- (3) $A \rightarrow B$ is a local complete intersection if and only if $\mathcal{L}_{B/A}^{LiSc} = \tau_{\geq -2} L_{B/A}$ is of perfect amplitude contained in $[-1, 0]$.

References and further reading

- André, M. : Méthode simpliciale en algèbra homologique et algèbra commutative. LNM, Springer (1967).
- Hovey, M. : Model categories. Mathematical surveys and monographs 63, American Math. Soc. (2007).
- Illusie, L. : Complexe cotangent et déformations - I. LNM, Springer (1971).
- Illusie, L. : Cotangent complex and deformations of torsors and group schemes. In *Toposes, Algebraic Geometry and Logic*, F.W. Lawvere (Ed), LNM, Springer (1972).
- Lichtenbaum, S. and Schlessinger, M : The cotangent complex of a morphism. Trans. American Math. Soc. 128 (1967).
- Majadas, J. and Rodicio, A.G. : Smoothness, regularity and complete intersection. London Math. Soc. Lecture Notes, Camb. Univ. Press (2010).

Quillen, D.G. : Homotopical algebra. LNM, Springer (1967).

Quillen, D.G. : On the (co-)homology of commutative rings. In *Applications of Categorical Algebra*, Proc. Symp. Pure Math. 17, Amer. Math. Soc. (1968).

Shukla, U. : Cohomologie des algèbres associatives. Ann. Sci. E.N.S. 78 (1961).

Stacksproject (A.J. de Jong et al) : <https://stacks.math.columbia.edu/>