

1 Derivations and Differentials

Leibniz's approach to differentiation had two closely related fundamental aspects. One was the use of first-order infinitesimals, and the other was the identification of certain formal properties satisfied by the differentiation operators. When formulated algebraically, these two aspects together give rise to an adjoint pair of functors – in which the functor defined via first-order infinitesimals is the right adjoint to the functor defined via Leibniz rules – that is the basis of a purely algebraic theory of derivations and differentials.

All rings in what follows are assumed to be commutative unless otherwise indicated.

1.1 Definition. Let $A \rightarrow B$ be a homomorphism of rings, and let M be a B -module. An **A -derivation on B with values in M** is an **A -linear** homomorphism $v : B \rightarrow M$ which satisfies the **Leibniz rule**, that is, $v(b_1 b_2) = b_1 v(b_2) + b_2 v(b_1)$.

All A -derivations on B with values in M form a B -module $Der_A(B, M)$ under 'pointwise' operations. That is, addition and scalar multiplication on $Der_A(B, M)$ are defined by $(v_1 + v_2)(x) = v_1(x) + v_2(x)$ and $(bv)(x) = b \cdot (v(x))$ for any $v, v_1, v_2 \in Der_A(B, M)$, $x \in M$ and $b \in B$.

1.2 It can be seen that the functor $Der_A(B, -) : B\text{-mod} \rightarrow B\text{-mod}$ is **B -linear** and **left-exact**. However, it is **not always exact**, as shown by the following example: Let $A = k$ be an arbitrary field, and let $B = k[\epsilon]/(\epsilon^2)$ as a k -algebra. The additive group A has the structure of a B -module as the quotient $j : B \rightarrow B/(\epsilon) = A$. Exercise: Describe the B -modules $Der_A(B, B)$ and $Der_A(B, A)$, making cases according to whether the characteristic of k is 2 or $\neq 2$. Show that $j_* : Der_A(B, B) \rightarrow Der_A(B, A)$ is not surjective, whatever be the characteristic of k .

1.3 The right derived functors of $Der_A(B, -)$.

As shown by Theorem 1.4, the functor $Der_A(B, -) : B\text{-mod} \rightarrow B\text{-mod}$ is representable, that is, there exists a B -module $\Omega_{B/A}$ together with a natural isomorphism of functors $Der_A(B, -) = Hom_{B\text{-mod}}(\Omega_{B/A}, -)$ defined by a universal derivation $d_{B/A} : B \rightarrow \Omega_{B/A}$. Hence the right derived functors of $Der_A(B, -)$ are identified with $Ext_B^i(\Omega_{B/A}, -)$, which are the right derived functors of $Hom_{B\text{-mod}}(\Omega_{B/A}, -)$. In particular, $Der_A(B, -)$ is exact if and only if $\Omega_{B/A}$ is a projective B -module.

1.4 Theorem. For any ring homomorphism $A \rightarrow B$, the corresponding B -linear functor $Der_A(B, -) : B\text{-mod} \rightarrow B\text{-mod}$ is **representable**. Equivalently (by Yoneda), there exists a B -module $\Omega_{B/A}$ together with an A -derivation $d_{B/A} : B \rightarrow \Omega_{B/A}$ such that any A -derivation $d : A \rightarrow M$ factors as $d = \varphi \circ d_{B/A}$ for a unique B -linear homomorphism $\varphi : \Omega_{B/A} \rightarrow M$.

The B -module $\Omega_{B/A}$ is called the **module of Kähler differentials** of $A \rightarrow B$, and the A -derivation $d_{B/A} : B \rightarrow \Omega_{B/A}$ is called the **universal derivation** on B .

over A . The pair $(\Omega_{B/A}, d_{B/A})$ is **uniquely unique** (means unique up to a unique isomorphism) by its universal property, provided it exists (which it always does).

1.5 Elementary properties. Constructions and various elementary property of Kähler differentials are given in standard textbooks in commutative algebra and algebraic geometry. We will take as known the properties proved in the Chapter 2 Section 8 of the standard textbook *Algebraic Geometry* by Hartshorne. The most important of these elementary properties for our immediate purpose are those related to changes of rings, which we now review. We now briefly recall some of this material.

1.6 Constructions of $(\Omega_{B/A}, d_{B/A})$.

(1) The free A -module F on the underlying set $|B|$ of B has a submodule R generated by all relations given by the definition of an A -derivation. The quotient of F by R gives as $(\Omega_{B/A}, d_{B/A})$.

(2) Let $I \subset B \otimes_A B$ denote the **diagonal ideal**, which is the kernel of the A -alg homomorphism $(\text{id}_B, \text{id}_B) : B \otimes_A B \rightarrow B$. It is generated by all elements of the form $1 \otimes b - b \otimes 1$ where $b \in B$. The ring $B \otimes_A B$ is naturally a left-right B -module, but the induced left and right B -module structures on I/I^2 coincide with each other. We can take $\Omega_{B/A} = I/I^2$ as a B -module, and define $d_{B/A} : B \rightarrow I/I^2$ by $b \mapsto (1 \otimes b - b \otimes 1)$.

1.7 Exercise: Finiteness of $\Omega_{B/A}$. If $f : A \rightarrow B$ is a finite type homomorphism where A is a noetherian ring, then show that $\Omega_{B/A}$ is a finite B -module.

1.8 Notation for adjoint functors. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say that (F, G) is an adjoint pair, in which F is left-adjoint to G (or G is right-adjoint to F) if for each $X \in \text{Ob } \mathcal{C}$ and $Y \in \text{Ob } \mathcal{D}$ we are given a bijection $\alpha_{X,Y} : \mathcal{C}(X, GY) \rightarrow \mathcal{D}(FX, Y)$ that is functorial in both X and Y (the obvious appropriate diagrams commute). Here, we have also used the notation $\mathcal{C}(-, -)$ for $\text{Hom}_{\mathcal{C}}(-, -)$ for any category $\mathcal{C}$. When α is understood as given, we omit it from the notation and simply write $\mathcal{C}(X, GY) = \mathcal{D}(FX, Y)$. Alternative notations for this adjoint pair are

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G \quad \text{or} \quad G : \mathcal{D} \rightleftarrows \mathcal{C} : F.$$

Both the above displayed notations combine the two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ into a single double-decker pair of mutually opposite arrows, with the convention that the left adjoint functor is always at the top and the right adjoint functor is always at the bottom of the pair $\rightleftarrows$ or $\rightleftarrows$, and the name of each functor is written close to its domain category.

1.9 The adjoint pair of functors $U : A\text{-alg} \rightleftarrows \text{Set} : P_A$.

Let A be any ring and S any set. Let $P_A(S) = A[S]$ denote the polynomial algebra

on A in the independent variables $[s]$ where $s \in S$. Associating $S \mapsto A[S]$, and associating the A -algebra homomorphism $A[S] \rightarrow A[T] : [s] \mapsto [f(s)]$ to a set map $f : S \rightarrow T$ defines a functor $P_A : \text{Set} \rightarrow A\text{-alg}$ that is left adjoint to the forgetful functor $U : A\text{-alg} \rightarrow \text{Set}$ that associates the underlying sets of elements and maps on these to A -algebras and their homomorphisms.

1.10 Differentials for a polynomial algebra. With notation as above, the universal pair $(\Omega_{A[S]/A}, d)$ can be so chosen that $\Omega_{A[S]/A}$ is the free $A[S]$ -module on the set of formal symbols ds , and $d : A[S] \rightarrow \Omega_{A[S]/A}$ is defined by the formula $d(f) = \sum_{s \in S} \partial f / \partial [s] ds$ which is a finite sum as only finitely many variables $[s]$ occur in any given $f \in A[S]$ even when the set S is infinite. With this definition of d , note that $d[s] = ds$.

Change of rings

1.11 Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms. Then the ring B becomes an A -algebra via f , the ring C becomes a B -algebra via g , and the ring C becomes a A -algebra via $g \circ f$. In geometric terms, we have morphisms $\text{Spec } C \rightarrow \text{Spec } B \rightarrow \text{Spec } A$ of affine schemes, and we have the corresponding adjoint pairs of direct image and inverse image functors between the categories of modules over these rings.

Consider the universal derivation $d_{C/A} : C \rightarrow \Omega_{C/A}$. The composite map $B \xrightarrow{g} C \xrightarrow{d_{C/B}} \Omega_{C/B}$ is an A -linear derivation on B with values in the B -module $\Omega_{C/B}$. Hence by the universal property of $(\Omega_{B/A}, d_{B/A})$, there exists a unique homomorphism $\alpha_{f,g} : \Omega_{B/A} \rightarrow \Omega_{C/A}$ of B -modules, such that the following square commutes.

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ \downarrow & & \downarrow \\ \Omega_{B/A} & \xrightarrow{\alpha_{f,g}} & \Omega_{C/A} \end{array}$$

The direct image functor $C\text{-mod} \rightarrow B\text{-mod}$ has as its left adjoint the pullback functor $C \otimes_B - : B\text{-mod} \rightarrow C\text{-mod}$. Hence the above B -module homomorphism $\Omega_{B/A} \rightarrow \Omega_{C/A}$ uniquely corresponds to a C -module homomorphism $1 \otimes \alpha_{f,g} : C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A}$. In terms of generators, any element $c \otimes d_{B/A}b \in C \otimes_B \Omega_{B/A}$ gets mapped under $\alpha_{f,g}$ to the element $cd_{C/A}g(b) \in \Omega_{C/B}$ (in simpler notation that suppresses B , C , f and g , this reads $cdb \mapsto cdb$).

The universal derivation $d_{C/B} : C \rightarrow \Omega_{C/B}$ is automatically A -linear. Hence by the universal property of $(\Omega_{C/A}, d_{C/A})$, there exists a unique homomorphism $\beta_{f,g} : \Omega_{C/A} \rightarrow \Omega_{C/B}$ of C -modules, such that the following square commutes.

$$\begin{array}{ccc} C & = & C \\ \downarrow & & \downarrow \\ \Omega_{C/A} & \xrightarrow{\beta_{f,g}} & \Omega_{C/B} \end{array}$$

In terms of generators, any element $d_{C/A} \in \Omega_{C/A}$ gets mapped under $\beta_{f,g}$ to the element $d_{C/B} \in \Omega_{C/B}$ (in simpler notation that suppresses B, C, f and g , this reads $dc \mapsto dc$).

1.12 The first Kähler sequence. The resulting sequence of C -modules

$$C \otimes_B \Omega_{B/A} \xrightarrow{1 \otimes \alpha_{f,g}} \Omega_{C/A} \xrightarrow{\beta_{f,g}} \Omega_{C/B} \rightarrow 0$$

is functorial in the data $(A \xrightarrow{f} B \xrightarrow{g} C)$ (Exercise: Make this statement precise by defining suitable categories and functors). This sequence is called the first Kähler sequence for $A \xrightarrow{f} B \xrightarrow{g} C$.

1.13 Proposition. Let $A \rightarrow B \rightarrow C$ be ring homomorphisms. The associated first Kähler sequence $C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$ is an exact sequence of C -modules.

1.14 The second Kähler sequence. (Also called the **co-normal sequence** or the **second fundamental sequence**.)

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms, such that $g : B \rightarrow C$ is surjective, and so $\Omega_{C/B} = 0$. Let $I = \ker(g)$. The map $d_{B/A} : B \rightarrow \Omega_{B/A}$ restricts to $I \subset B$ to define a map $d_{B/A}|_I : I \rightarrow \Omega_{B/A}$. Applying $C \otimes_B -$ to the above map, and noting that we have a functorial isomorphism $C \otimes_B M = M/IM$ for B -module, we have an induced homomorphism of C -modules $\gamma_{f,g} : I/I^2 \rightarrow \Omega_{B/A}/I\Omega_{B/A} = C \otimes_B \Omega_{B/A}$. This produces a sequence of C -modules

$$I/I^2 \xrightarrow{\gamma_{f,g}} C \otimes_B \Omega_{B/A} \xrightarrow{1 \otimes \alpha_{f,g}} \Omega_{C/A} \xrightarrow{\beta_{f,g}} 0$$

that is functorial in the data $(A \xrightarrow{f} B \xrightarrow{g} C)$. This is called the second Kähler sequence for $A \rightarrow B \rightarrow C$ when $B \rightarrow C$ is surjective with kernel I . In the special case when B is a polynomial algebra over A , the second Kähler sequence is also called the **Jacobi-Zariski sequence**.

1.15 Proposition. Let $A \rightarrow B \rightarrow C$ be ring homomorphisms, where $B \rightarrow C$ is surjective with kernel $I \subset B$. Its functorially associated second Kähler sequence $I/I^2 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow 0$ is an exact sequence of C -modules.

1.16 Exercise: Examples. (1) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms such that $C = B[X]$, a polynomial algebra in a set X of independent variables over B . Show that in the corresponding first Kähler sequence is short-exact, that is, the C -linear homomorphism $1 \otimes \alpha_{f,g} : C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A}$ is injective.

(2) Let $A = k$ be any field. Let $C = k[t]$ be a polynomial algebra over k in a single variable, and let $B \subset C$ be the sub- k -algebra $k[t^2, t^3] \subset k[t]$. In the first Kähler

sequence for $A \rightarrow B \rightarrow C$ in this example, determine the kernel of $C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A}$.

(3) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be ring homomorphisms such that $g : B \rightarrow C$ is surjective, and the A -algebra $g \circ f : A \rightarrow C$ is isomorphic to a polynomial A -algebra $A[X]$ for a set X . Show that the corresponding second Kähler sequence is short-exact, that is, the C -linear homomorphism $I/I^2 \rightarrow C \otimes_B \Omega_{B/A}$ is injective.

(4) Determine the kernel of $I/I^2 \rightarrow C \otimes_B \Omega_{B/A}$ in the second Kähler sequence when A is a field k , B a polynomial ring $k[x, y]$ and $C = B/I$ where $I = (x^2 - y^3)$.

(5) Determine the kernel of $I/I^2 \rightarrow C \otimes_B \Omega_{B/A}$ when A is a field k , B is the polynomial ring $k[x]$, and $I = (x^2)$. How does it depend on the characteristic of k ?

1.17 Commutative square of ring homomorphisms. Let $f : A \rightarrow B$, $f' : A' \rightarrow B'$, $g : A \rightarrow A'$ and $h : B \rightarrow B'$ be ring homomorphisms such that $h \circ f = f' \circ g : A \rightarrow B'$, making B' into the same A -algebra either way. The composite map $d_{B'/A'} \circ h : B \rightarrow \Omega_{B'/A'}$ is an A -derivation on B . Hence by the universal property of $(\Omega_{B/A}, d_{B/A})$, there exists a unique B -linear homomorphism $h' : \Omega_{B/A} \rightarrow \Omega_{B'/A'}$ such that the following diagram commutes.

$$\begin{array}{ccc} \Omega_{B/A} & \xrightarrow{h'} & \Omega_{B'/A'} \\ d_{B/A} \uparrow & & \uparrow d_{B'/A'} \\ B & \xrightarrow{h} & B' \\ f \uparrow & & \uparrow f' \\ A & \xrightarrow{g} & A' \end{array}$$

1.18 Base change. Let $f : A \rightarrow B$ and $g : A \rightarrow C$ be ring homomorphisms. Let $B \otimes_A C$ together with ring homomorphisms $h_1 : B \rightarrow B \otimes_A C : b \mapsto b \otimes 1$ and $h_2 : C \rightarrow B \otimes_A C : c \mapsto 1 \otimes c$. Note that $h_1 \circ f = h_2 \circ g : A \rightarrow B \otimes_A C$, and so this is a special case of the statement 1.17, where the lower commutative square of the above diagram is the following pushout diagram (co-cartesian square) of rings.

$$\begin{array}{ccc} B & \xrightarrow{h_1} & B \otimes_A C \\ f \uparrow & & \uparrow h_2 \\ A & \xrightarrow{g} & C \end{array}$$

Then we have the following.

Proposition. (1) The Kähler differentials of $B \otimes_A C$ over B are given by

$$\Omega_{B \otimes_A C / C} = \Omega_{B/A} \otimes_B (B \otimes_A C) = \Omega_{B/A} \otimes_A C$$

with universal derivation equal to the map $d_{B/A} \otimes_A 1$ that sends $b \otimes_A c \mapsto (d_{B/A} b) \otimes_A c$, and the map $h' : \Omega_{B/A} \rightarrow \Omega_{B'/A'}$ of statement 1.17 corresponds to the map $\Omega_{B/A} \rightarrow \Omega_{B/A} \otimes_A C$ that sends $\beta \mapsto \beta \otimes_A 1$.

(2) A similar statement holds in which $f : A \rightarrow B$ and $g : A \rightarrow C$ are interchanged in the above statement (1).

(3) The Kähler differentials of $B \otimes_A C$ over A are given by

$$\Omega_{B \otimes_A C/A} = (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A}) \quad \text{with} \quad d_{B \otimes_A C/A} = d_{B/A} \otimes 1 + 1 \otimes d_{C/A}$$

1.19 Derivation sequences (dual Kähler sequences). Let $A \rightarrow B \rightarrow C$ be ring homomorphisms and let N be a C -module. Applying $\text{Hom}_C(-, N)$ to the first Kähler sequence produces the following functorial left-exact sequence (the first derivation sequence).

$$0 \rightarrow \text{Der}_B(C, N) \rightarrow \text{Der}_A(C, N) \rightarrow \text{Der}_A(B, N)$$

Now suppose that moreover $C = B/I$ for an ideal I , and $B \rightarrow C$ is the quotient map. Then $\text{Der}_B(C, N) = 0$, and applying $\text{Hom}_C(-, N)$ to the second Kähler sequence produces the following functorial left-exact sequence (the second derivation sequence).

$$0 \rightarrow \text{Der}_A(C, N) \rightarrow \text{Der}_A(B, N) \rightarrow \text{Hom}_C(I/I^2, N)$$

Comparison with C^∞ -manifolds

Now that we have set up algebraic derivations and universal Kähler differentials, we can take a look at the category of C^∞ -manifolds from this ‘algebraic’ perspective of derivations and differentials. To begin with, the usual absolute value on $\mathbb{R}$ together with Cauchy’s ϵ - δ method are used to define the topology on $\mathbb{R}^n$ and to define the $\mathbb{R}$ -algebra $C^\infty(U)$ of $\mathbb{R}$ -valued functions on open subsets U in $\mathbb{R}^n$. Next, C^∞ -manifolds (X, C_X^∞) are defined as locally ringed spaces of algebras of $\mathbb{R}$ -valued functions, locally modeled over opens in $\mathbb{R}^n$. Morphisms between manifolds $(X, C_X^\infty) \rightarrow (Y, C_Y^\infty)$ are defined as continuous maps $f : X \rightarrow Y$ that pull-back germs of C^∞ -functions on Y to germs of C^∞ -functions on X . Till this point, the algebraic theory of derivations does not play any role. The global theory of derivations is based on the following local facts (proofs left to the reader).

1.20 Lemma. (1) Let (x_i) be cartesian coordinates on $\mathbb{R}^n$. Let $P = (a_1, \dots, a_n) \in \mathbb{R}^n$ and let U be any open ball containing P . Let f be a C^∞ -function on U . Then there exist C^∞ -functions g_i on U for $1 \leq i \leq n$ such that $f = f(P) + \sum_i (x_i - a_i)g_i$.

(2) Let $C_{\mathbb{R}^n, P}^\infty$ denote the local ring of germs of C^∞ -functions on $\mathbb{R}^n$ at P . Then its maximal ideal $\mathfrak{m}_P$ is generated by the n -elements $x_i - a_i$. The corresponding elements $x_i - a_i + \mathfrak{m}_P^2$ form a basis of the $\mathbb{R}$ -vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.

(3) There exist $b_i \in \mathbb{R}$ and $h_{ij} \in C^\infty(U)$ for $1 \leq i, j \leq n$, such that $f = f(P) + \sum_i (x_i - a_i)b_i + \sum_{i,j} (x_i - a_i)(x_j - a_j)h_{ij}$.

(4) For any open subset W in $\mathbb{R}^n$ (which need not be a ball), the $C^\infty(W)$ -module $\text{Der}_{\mathbb{R}}(C^\infty(W), C^\infty(W))$ is free of rank n with basis $\partial_i = \partial/\partial x_i$, where x_i are

cartesian coordinates on $\mathbb{R}^n$. For any $v \in \text{Der}_{\mathbb{R}}(C^\infty(W), C^\infty(W))$, we have $v = \sum_i v(x_i) \partial/\partial x_i$. In particular, each $v \in \text{Der}_{\mathbb{R}}(C^\infty(W), C^\infty(W))$ is a **locally defined operator**, that is, if $f, g \in C^\infty(W)$ and $V \subset W$ is open such that $f|_V = g|_V$ then $v(f)|_V = v(g)|_V$.

The first major difference. In the algebraic theory, the functor $(X/S) \mapsto (\Omega_{X/S}, d_{X/S} : \mathcal{O}_X \rightarrow \Omega_{X/S})$ is the primary differential object, and the rest are made from it. For C^∞ -manifolds, the primary object is the tangent sheaf Θ_X of a manifold X , and its local action as $\mathbb{R}$ -derivations $\Theta_X \times C_X^\infty \rightarrow C_X^\infty$. To construct this, for an open subset $W \subset \mathbb{R}^n$, we define

$$\Theta(W) = \text{Der}_{\mathbb{R}}(C^\infty(W), C^\infty(W))$$

By the above lemma, this is a locally free $C^\infty(W)$ -module of rank n with free basis given by the operators $\partial/\partial x_i$. Again using the above lemma, these naturally glue together to define a locally free sheaf Θ_X of C_X^∞ -modules on any manifold X , together with its natural local action as $\mathbb{R}$ -derivations on C_X^∞ . Then the sheaf of 1-forms $\Theta_X^\vee$ with exterior derivative $d_X : C_X^\infty \rightarrow \Theta_X^\vee$ is the dual of the above. This is again locally free of rank n , with local basis dx_i dual to $\partial/\partial x_i$ in a local coordinate chart. This sheaf $\Theta_X^\vee$ together with the exterior derivative $d_X : C_X^\infty \rightarrow \Theta_X^\vee$ is **not** in general the sheaf of relative Kähler differentials for the C^∞ -morphism $X \rightarrow \text{Spec } \mathbb{R}$, which would be ‘too large’.

Example. Let $X = (-1, 1)$ be regarded as a C^∞ -manifold, with coordinate x . Let $K = \mathbb{R}((x))$ denote the field of formal meromorphic series in x . Then x and $e^x = \sum_{n \geq 0} x^n/n!$ are algebraically independent in K over $\mathbb{R}$. Cartier’s theorem ?? implies that the first Kähler sequence of $\mathbb{R} \rightarrow \mathbb{R}(x, e^x) \rightarrow K$ is short-exact, and so the elements $d_{K/\mathbb{R}}(e^x)$ and $d_{K/\mathbb{R}}x$ are K -linearly independent in $\Omega_{\mathbb{R}((x))/\mathbb{R}}$. Consider the $\mathbb{R}$ -algebra homomorphism $C^\infty(X) \rightarrow K$ that sends any function f to its formal Taylor series $\sum_{n \geq 0} f^{(n)}(0)x^n/n!$ at 0. Hence the $d_{K/\mathbb{R}}(e^x)$ is not a multiple of $d_{K/\mathbb{R}}x$ in $\Omega_{C^\infty(X)/\mathbb{R}}$. However, as sections of $\Theta_X^\vee$ we have $d_X(e^x) = e^x d_X x$, which illustrates the phenomenon described above.

The second major difference. In the algebraic category, we have relative Kähler differentials $(\Omega_{X/S}, d_{X/S})$ for any morphism $X \rightarrow S$ of schemes. In the C^∞ -category, we can define relative differentials only for morphisms $\varphi : X \rightarrow Y$ which have a locally constant Jacobian rank. In the locally constant rank case, these relative differentials again define a locally free sheaf of local rank equal to $\dim(X) - r$ where r is the local relative rank. But there is **no** good definition of relative differentials when the local rank is not constant.

The third major difference. For morphisms $\phi : X \rightarrow Y$ which are submersive (means the Jacobian rank of ϕ at a point $x \in X$ equals $\dim_{\phi(x)} Y$), we have the implicit function theorem. Such a theorem does not hold in the algebraic category for the analog of submersions (which are called ‘smooth morphisms’ in algebraic geometry, leading to an unfortunate clash of terminologies which can sometimes cause confusion). More generally, C^∞ -morphisms of locally constant rank have an easy to describe nice local structure, which is exploited in the study of C^∞ -manifolds, Lie groups, etc, while the corresponding algebraic theory (for once!) is more difficult.

The fourth major difference. Unlike in schemes, arbitrary fiber products **do not exist** in C^∞ -manifolds. There are two sufficient conditions for a fiber $X \times_S T$ product to exist: (1) When the morphism $X \rightarrow S$ is smooth, and (2) when the base-change morphism $T \rightarrow S$ is smooth. In the case (2), if the morphism $X \rightarrow S$ is of locally constant Jacobian rank, then so is the base-change $X_T = X \times_S T \rightarrow T$. In these cases, the relative tangent or cotangent bundles for $X \rightarrow S$ and for $X_T \rightarrow T$ exist, and have the expected behaviour under base changes.

The fifth major difference. This final remark will become clear when we look at cotangent complexes. The algebraic theory over schemes is an incomplete story for arbitrary morphisms of schemes, and the cotangent complexes give us the missing ‘higher’ or ‘derived’ picture. There is no need for an analog of cotangent complexes beyond $(\Theta_X^\vee, d_X)$ in the C^∞ -theory, as the C^∞ -theory considers relative differentials bundles (or vertical tangent bundles) only for smooth manifolds and constant rank morphisms, where there is no missing ‘higher’ picture.