Commutative Algebra, Problem Set 9

(1) Let I be an irreducible ideal in a ring A. Show that the following are equivalent.
   (a) I is primary.
   (b) For every multiplicative set S in A, $(S^{-1}I)^c = (I : x)$ for some $x \in S$.
   (c) The sequence $(I : x^n)$ stabilizes for every x in A.

(2) If I is a radical ideal, show that I has no embedded prime ideals.

(3) Let $A = k[X, Y, Z]$ be a polynomial ring in 3 variables over a field k. Consider the three ideals: $P_1 = (X, Y)$, $P_2 = (X, Z)$ and $\mathfrak{m} = (X, Y, Z)$. Let $I = P_1 P_2$. Show that $I = P_1 \cap P_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of I. Determine the isolated and embedded primes.

(4) Let I be a decomposable ideal in a ring A. Let P be a maximal element in the set of ideals of the form $(I : x)$ where $x \in A$ and $x \notin I$. Show that P is an associate prime of I.

(5) Let I be an ideal in a Noetherian ring A. Consider two minimal decompositions of I as intersections of irreducible ideals: $I = \cap_{i=1}^{r} J_i = \cap_{j=1}^{s} K_j$.
   Show that $r = s$ and (possibly after reindexing) $\sqrt{J_i} = \sqrt{K_i}$ for all i.
   Here the meaning of “minimal” is identical to the meaning of minimal primary decomposition.

(6) Let $A = k[X, Y, Z]$ be a polynomial ring in 3 variables over a field k. Let $I = (XY, X - YZ)$, $Q_1 = (X, Z)$ and $Q_2 = (Y^2, X - YZ)$. Show that $I = Q_1 \cap Q_2$ is a minimal primary decomposition.

(7) Let A be a ring and let P be a prime ideal. Denote by $P^{(n)}$ the n-th symbolic power of P. Show that:
   (a) If P^n is decomposable, then $P^{(n)}$ is its P-primary component.
   (b) If $P^{(n)} P^{(m)}$ is decomposable, then $P^{(m+n)}$ is its P-primary component.

(8) Let A be a noetherian ring. Let $\mathfrak{m} \subset A$ be a maximal ideal. The following are equivalent for an ideal Q:
   (a) Q is $\mathfrak{m}$-primary.
   (b) $\sqrt{Q} = \mathfrak{m}$.
   (c) $\mathfrak{m}^n \subset Q \subset \mathfrak{m}$ for some $n \geq 1$.

(9) Show that in a noetherian ring every ideal contains a power of its radical.

(10) Let $(A, \mathfrak{m})$ be a noetherian local domain of dimension 1. Show that
   (a) if I is a nonzero ideal then I is $\mathfrak{m}$-primary and $\mathfrak{m}^n \subset I$ for some $n \geq 1$.
   (b) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all $n \geq 0$.

(11) Let I, J be two ideals in a ring A such that $\sqrt{I}$ and $\sqrt{J}$ are coprime. Show that I and J are also coprime.