Commutative Algebra, Problem Set 8

(1) Let $f : A \to B$ be an integral homomorphism of rings (f need not be injective). This means that B is integral over $f(A)$. Show that the induced map of topological spaces $f^\star : Spec\ A \to Spec\ B$ is closed.

(2) Let A be a subring of B and B is integral over A. Suppose that $f : A \to K$ is a homomorphism into an algebraically closed field K. Show that f can be extended to B.

(3) Let $A \subset B$ be an integral extension of rings and suppose that either A or B has finite dimension. Show that A and B have same dimension.

(4) Suppose that $A \subset B$ is an extension of rings for which the going-down theorem holds. Let P be a prime in B and let $Q = P \cap A$. Show that $\dim(B_P) \geq \dim(A_Q)$.

(5) Suppose that $A \subset B$ is an integral extension of rings. Let P be a prime in B and let $Q = P \cap A$. Show that $\dim(B_P) \leq \dim(A_Q)$.

(6) *Integral closure of ideals*:

Let A be a domain and let I be an ideal. The integral closure of I in A is defined to be the set of elements $a \in A$ satisfying an equation of the form:

$$a^n + r_1 a^{n-1} + \ldots + r_n = 0, \text{ where } r_j \in I^j, \text{ the } j\text{−th power of } I.$$

Show that a is integral over I if and only if there is a finitely generated A-module N, not annihilated by any element of A such that $aN \subset IN$. Use this to show that the integral closure of I in A is an ideal.

(7) If A is a domain, show that every radical ideal is integrally closed in A.

(8) Let $f : A \to B$ be an inclusion of rings. Let $f^\star : Spec\ B \to Spec\ A$ be the associated map of topological spaces. Consider the following statements:

(a) $f^\star$ is a closed map.

(b) f has the going-up property.

(c) Let $P \subset B$ be a prime ideal and $Q = P \cap A$. Then $f^\star : Spec\ (B/P) \to Spec\ (A/Q)$ is surjective.

Prove that $(a) \Rightarrow (b) \Leftrightarrow (c)$.

(9) Let $f : A \to B$ be an inclusion of rings. Let $f^\star : Spec\ B \to Spec\ A$ be the associated map of topological spaces. Show that the following statements are equivalent:

(a) f has the going-down property.

(b) Let $P \subset B$ be a prime ideal and $Q = P \cap A$. Then $f^\star : Spec\ (B_P) \to Spec\ (A_Q)$ is surjective.

(10) Let A be a ring and let G be a finite group of automorphisms of A. Let A^G be the ring of invariants. Show that A is integral over A^G.

(11) (In the situation of the previous problem) Let Q be a prime ideal of A^G. Let F be the set of primes of A that contract to Q. If $g \in G$ and $P \in F$, show that $g(P) \in F$. Hence G acts on F. Show that this action of G on F is transitive. Conclude that F is finite.

(12) Let A be an integrally closed domain and let K be its quotient field. Let L be a finite Galois extension of K and let G be the Galois group of the extension. Finally let B be the integral closure of A in L.

Show that $g \in G \Rightarrow g(B) = B$ and that $A = B^G$.

Use this to show that $Spec\ B \to Spec\ A$ has *finite fibers* (meaning that given a prime $Q \subset A$, there are only a finitely many primes $P \subset B$ such that $P \cap A = Q$).

(13) Take the same set up as in the previous problem, but assume only that $K \subset L$ is a finite extension. Show that $Spec\ B \to Spec\ A$ has finite fibers in this case also.

Hint: consider two cases: L is separable over K and L is purely inseparable over K.