Commutative Algebra, Problem Set 7

(1) Show that a UFD is integrally closed.

(2) Suppose that $A \subset B$ is an integral extension of rings and that B is a finitely generated algebra over A. Show that B is a finitely generated A-module.

(3) Let $A \subset B$ be an integral extension. If $a \in A$ is a unit in B show that a is a unit in A. Is this true without the assumption of integrality?

(4) Let $A \subset B$ be an integral extension of integral domains. Show that A is a field if and only if B is a field.

(5) Let $A \subset B$ be such that $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

(6) Let A be an integral domain with the field of fractions k. An element $x \in k$ is said to be *almost integral* if there exists an element $0 \neq a \in A$ such that $ax^n \in A$ for all $n > 0$.

Show that if $x \in k$ is integral then it is almost integral. Prove the converse when A is noetherian.

(7) Let $n \in \mathbb{Z}$. Find the integral closure of $A = \mathbb{Z}[\sqrt{n}]$ following the steps below:
  (a) Reduce to the case that n is square-free.
  (b) Show that $\sqrt{n}$ is integral over $\mathbb{Z}$. Conclude that the integral closure of A is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{n}]$. Denote by B the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{n}]$.
  (c) Let $\alpha = a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$. Minimal polynomial of α over $\mathbb{Q}$ is $x^2 - Trace(\alpha) + Norm(\alpha)$, where $Trace(\alpha) = 2a$ and $Norm(\alpha) = a^2 - b^2n$. Thus $\alpha \in B$ if and only if $Trace(\alpha)$ and $Norm(\alpha)$ are integers.
  (d) Show that if $\alpha \in B$ then $a \in \frac{1}{2}\mathbb{Z}$.
  If $a = 0$, show that $\alpha \in B$ if and only if $b \in \mathbb{Z}$.
  If $a = 1/2$ and $\alpha \in B$, show that $b \in \frac{1}{2}\mathbb{Z}$.
  Thus, repeatedly subtracting integers and multiples of $\sqrt{n}$ from α, we may assume $a = b = 0$ or $a = b = 1/2$.
  (e) Finally show that the integral closure of A is $\mathbb{Z}[\sqrt{n}]$ if $n \not\equiv 1 (\text{mod } 4)$ and is $\mathbb{Z}[1/2 + 1/2\sqrt{n}]$ if $n \equiv 1 (\text{mod } 4)$.