Commutative Algebra, Problem Set 6

(1) Let $(A, \mathfrak{m}) \subset (B, \mathfrak{n})$ be local rings. Show that $\mathfrak{m} \subset \mathfrak{n} \Leftrightarrow \mathfrak{m} = \mathfrak{n} \cap A$. If this is the case, we say B *dominates* A.

(2) Let A be an integral domain with quotient field k. We say that A is a *valuation ring of* k if for every $0 \neq x \in k$, either $x \in A$ or $x^{-1} \in A$.
   (a) Show that a valuation ring is local.
   (b) If $A \subset B \subset k$ and A is a valuation ring of k, then B is a valuation ring of k.
   (c) Show that A is a valuation ring $\Leftrightarrow$ for any two ideals $I, J \subset A$, either $I \subset J$ or $J \subset I$.

(3) Let k be an algebraically closed field.
   (a) Describe the topological space *Spec* $k[X]$.
   (b) Describe the subspace max-*Spec* $k[X]$.
   (c) What are the points in *Spec* $k[X]$ that are not in max-*Spec* $k[X]$.

(4) Consider the subring R of $k(X)$ where k is a field defined as: $R = \{\frac{f(X)}{g(X)} | g(0) \neq 0\}$.
   Show that R is a local ring with maximal ideal generated by X. Is R a valuation ring?

(5) Let R be an integral domain containing a field k. Further assume that dimension of R as a k-vector space is finite. Show that R is a field.

(6) Let $A \subset B \subset C$ be rings. Suppose that C is a finitely generated A-algebra and a finitely generated B-module and A is noetherian. Show that B is a finitely generated A-algebra.
   Is this true if C is not a finitely generated B-module? What if A is not noetherian?

(7) Let k be a field and let $R = k[X, Y]/(X^2, XY)$. Let A be the localization of R at the maximal ideal (x, y) of R. Denote the unique maximal ideal of A by $\mathfrak{m}$. Show that every element of $\mathfrak{m}$ is a zero divisor and that A has dimension 1.

(8) Let $(R, \mathfrak{m}, k)$ be a noetherian local domain such that $\mathfrak{m}$ is a principal ideal. Consider the function $f(n) = dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, $n \geq 1$. Here dim_k is the dimension as k-vector space. Show that $f(n)$ is eventually constant, and moreover this constant is either 1 or 0.