Commutative Algebra, Problem Set 5

(1) Let R be a ring such that for every maximal ideal $P \subset R$ the local ring R_P is noetherian. Further assume that for every $a \in R$, there are only finitely many maximal ideals containing x. Show that R is noetherian.

(2) Let A be a ring and let $S \subset A$ be a multiplicative set. Let $I \subset A$ be an ideal. Show the following:

(a) $I^{ec} = \cup_{s \in S}(I : s)$. So $I^e = S^{-1}A$ if and only if $I \cap S \neq \emptyset$.

(b) I is a contracted ideal if and only if no element of S is a zero divisor in A/I.

(c) The operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.

(3) Let $A \to B$ be a ring homomorphism and let $P \subset A$ be a prime ideal. Show that P is a contraction of a prime ideal of B if and only if $P^{ec} = P$.

(4) Let $S \subset A$ and let M be a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

(5) Consider the ring R of all sequences with entries in $\mathbb{Z}/2\mathbb{Z}$ that are eventually constant (with coordinate wise addition and multiplication). Show that R is not noetherian. Is it an integral domain? Show that every prime ideal of R is maximal. (Hint: observe that it is a Boolean ring)

Consider the ideals

$$M_i = \{\{a_n\} | a_i = 0\} \text{ and } M_\infty = \{\{a_n\} | a_i = 0 \text{ for large } n\}$$

Show that each M_i and M_∞ are minimal primes of R. So a nonnoetherian ring, in general, doesn't have only finitely many minimal primes.

(6) Let $f : R \to S$ be a ring homomorphism. Show that the induced map of topological spaces $f^\sharp : \mathit{Spec}\ S \to \mathit{Spec}\ R$ is continuous.

(7) If $S \subset A$ is multiplicative, show that $\mathit{Spec}(A_S)$ is homeomorphic to the subspace $\{P \in \mathit{Spec}\ A | A \cap S = \emptyset\}$.

(8) Let R be a ring and let $P \subset R$ be a prime ideal. Show that $V(P)$ is an irreducible closed subset of $\mathit{Spec}\ R$. Conversely, show that any irreducible closed subset of $\mathit{Spec}\ R$ can be written in the form $V(P)$ for a prime ideal P in R.

(9) If I is an ideal of A, then $\mathit{Spec}(A/I)$ is homeomorphic to the closed subset $V(I)$ of $\mathit{Spec} A$.

(10) Let $I \subset R$ be an ideal. Show that $V(I) = \emptyset \Leftrightarrow I = R$.

(11) Show that $\mathit{Spec}\ A$ is quasi-compact for any ring A.

(12) Let $f : A \to B$ be a ring homomorphism and let $f^\sharp : \mathit{Spec}\ B \to \mathit{Spec}\ A$ be the associated continuous map of topological spaces. Show that:

(a) Every prime of A is a contracted ideal $\Leftrightarrow f^\sharp$ is surjective.

(b) Every prime of B is an extended ideal $\Rightarrow f^\sharp$ is injective.

Is the converse to (b) true?

(13) If A and B are rings, then show that $\mathit{Spec}(A \times B)$ can be identified with disjoint union $\mathit{Spec}\ A \coprod \mathit{Spec}\ B$, with both closed and open in $\mathit{Spec}(A \times B)$.

(14) Show that the (Krull) dimension of the polynomial ring $k[X]$ over a field k is 1.

For the next problem assume that the dimension of the polynomial ring $k[X_1, \ldots, X_n]$ over a field k is n.

(15) Find the dimension of the following rings:
  (a) $R_1 = k[X, Y]/(XY)$
  (b) $R_2 = k[X, Y]/(X^2, XY)$
  (c) $R_3 = R_1/(x)$
  (d) $R_4 = \mathbb{R}[X, Y]/(X^2 + 1)$
  (e) $R_5 = \mathbb{R}[X, Y]_{(X^2+1)}$
  (f) $R_6 = \mathbb{Z}_{p\mathbb{Z}}$, where p is a prime
  (g) $R_7 = \mathbb{Z}_S$, where $S = \{1, 3, 9, 27, \ldots\}$