Commutative Algebra, Problem Set 4

(1) Let $S \subset A$ be multiplicative. Show that the canonical map $A \to A_S$ is an isomorphism if and only if S consists of units.

(2) Let $a \in A$ and $S = \{1, a, a^2, \ldots\}$. Let $A[X]$ be the polynomial ring in one variable over A. Show that the localized ring A_S is isomorphic to $A[X]/(aX - 1)$.

(3) Describe all intermediate rings $\mathbb{Z} \subset R \subset \mathbb{Q}$. Hint: realize them as localizations of $\mathbb{Z}$ at appropriate multiplicative sets.

(4) Let R and S be two rings.
  (a) Describe the prime ideals of the product ring $R \times S$.
  (b) Prove that R and S are localizations of $R \times S$.
  (c) Show that $(R \times S)_{P \times S} \cong R_P$ for a prime ideal P in R.

(5) Give an example of a ring A and a multiplicative set S in A to show that the following is **not** an equivalence relation on $A \times S$: $(a, s) \sim (b, t)$ if $at = bs$.

(6) Let A be a ring and let $S \subset A$ be multiplicative. Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules. Show that $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is an exact sequence of $S^{-1}A$-modules.

(7) Suppose that A_P has no nonzero nilpotent elements for every prime P in A. Show that A has no nonzero nilpotent elements.

(8) Find an example of a ring A such that A_P is an integral domain for every prime ideal P, but A is not an integral domain.

(9) Find a ring A and distinct multiplicative sets S and T such that $S^{-1}A = T^{-1}A$.

(10) Let $S \subset A$ be multiplicative. Define $T = \{t \in A | at \in S \text{ for some } a \in A\}$. Show that T is a maximal element in the family of multiplicative sets T such that $S^{-1}A = T^{-1}A$. T is called the *saturation* of S.

(11) A multiplicative set S is called *saturated* if $ab \in S \Leftrightarrow a, b \in S$. Show that a multiplicative set
  (a) S is saturated if and only its saturation is S.
  (b) S is saturated if and only if the complement of S is a union of prime ideals.

(12) Let $R = \mathbb{Z}$ and let $S = \mathbb{Z} - \{0\}$. Consider the R-modules $M_n = \mathbb{Z}/n\mathbb{Z}$, for $n \geq 2$. Show that $S^{-1}M_n = 0$ for all n. On the other hand, show that $S^{-1}(\prod_n M_n) \neq 0$.

(13) Let $P \subset R$ be a prime ideal and let $n \geq 1$. Define the *n-th symbolic power*, $P^{(n)}$, of P as: $P^{(n)} = \{r \in R | sr \in P^n \text{ for some } s \in R \setminus P\}$.
  (a) Show that $P = P^{(1)}$ and $P \subset P^{(n)}$ for all $n \geq 2$.
  (b) Show that $P^{(n)} = f^{-1}(P^n R_P)$, where R_P is the localization of R at the prime ideal P and $f : R \to R_P$ is the natural map.
  (c) Show that for each $n \geq 1$, $P^{(n)}$ is a primary ideal with radical P.
  (d) If P is maximal show that $P^n = P^{(n)}$ for all $n \geq 1$.
  (e) Show that in general $P \neq P^{(n)}$ for $n \geq 2$ using the example below:
  Let k be a field and let $R = k[X, Y, Z]/(XZ - Y^2) = k[x, y, z]$. Consider the ideal $P = (x, y)$. Show that P is prime and that $P^n \neq P^{(n)}$ for $n \geq 2$.