Commutative Algebra, Problem Set 3

(1) If a ring R is artinian, then is the polynomial ring $R[X]$ also artininan?
(2) If $R[X]$ is noetherian, then is R noetherian?
(3) Is $\mathbb{Q}/\mathbb{Z}$ artinian as a $\mathbb{Z}$-module?
(4) Let $(R, \mathfrak{m})$ be a local ring. Prove that either $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all n or $\mathfrak{m}^n = 0$ for some n. Prove also that in the second case, R is artinian.
(5) Show that in an artinian ring every prime ideal is maximal.
(6) Show that an artinian ring has only finitely many maximal ideals.
(7) Let R be a noetherian ring and let $I \subset R$ be a *radical* ideal (i.e. $I = \sqrt{I}$). Show that I is a finite intersection of prime ideals.
(8) Let I be a radical ideal in a ring R. Let $I = P_1 \cap P_2 \cap \ldots \cap P_n$ be a minimal representation of I as an intersection of prime ideals. This means that no P_i can be omitted from the above representation. Then show that each P_i is a minimal prime of I (that is: minimal among all primes containing I). Further show that this representation of I as an intersection of primes is unique (with minimality condition).
(9) Let $I \subset R$ be a proper ideal. We say that I is *primary* if the following condition holds:

let $x, y \in R$; if $xy \in I$, then either $x \in I$ or $y^n \in I$ for some $n \geq 1$.

Now let $I \subset R$ be an ideal. Show that I is primary if and only if all the zero divisors of R/I are nilpotent.
(10) What are the primary ideals of $\mathbb{Z}$?
(11) Let k be a field and let $R = k[X, Y]$. Consider the ideals $I_1 = (X^n, Y^m)$ where n, m are some positive integers and $I_2 = (XY, Y^2)$. Are these ideals primary?
(12) Let $R = k[X, Y, Z]/(XY - Z^2)$. Denote the residues of X, Y, Z by x, y, z, so that $R = k[x, y, z]$. Let $I = (x, z)$ be an ideal of R. Show that I is prime and I^2 is not primary.
(13) Show that the radical of a primary ideal is prime. Is this true without the primary assumption?
(14) Let $I \subset R$. If the radical of I is a maxima ideal, show that I is primary. If the radical of I is a prime ideal, then is it true that I is primary? If I is primary, then is the radical of I maximal?