Commutative Algebra, Problem Set 2

(1) Show that a ring R is noetherian $\Leftrightarrow$ every ideal of R is finitely generated $\Leftrightarrow$ every prime ideal of R is finitely generated.

(2) Let R be a ring and let M be a module. Assume that M has a composition series. Show that the length of a composition series of M is independent of the series chosen.

(3) Show that $l(M) < \infty \Leftrightarrow M$ is an artinian and a noetherian module.

(4) Suppose that $0 \to M_1 \to M_2 \ldots \to M_r \to 0$ is an exact sequence of R-modules and each M_i has finite length. Show that $\sum_{i=1}^{r}(-1)^i l(M_i) = 0$.

(5) Let $(R, \mathfrak{m})$ be a local ring such that $\mathfrak{m}$ is principal and $\cap \mathfrak{m}^n = 0$. Show that R is noetherian and every nonzero ideal of R is a power of $\mathfrak{m}$.

(6) (a) Let k be a field and let R be the subring of $k[x, y]$ defined by $R = \{a + xf | a \in k,\ f \in k[x, y]\}$. Show that R is not noetherian, by finding an ideal that is not finitely generated.

(b) Let R be a subring of a noetherian ring S. Suppose that R is a *summand* of S as an R-module (that is, there is a homomorphism $S \to R$ of R-modules fixing every element of R). Show that R is noetherian.

(7) Let R be a noetherian ring and let I be an ideal. Consider the family $\mathfrak{F}$ of prime ideals containing I. Show that $\mathfrak{F}$ has only finitely many minimal elements. These are called the *minimal primes* of I.

Hint: Consider the family of ideals I that violate this.

(8) Let R be a ring and let A, B, C be R-modules. Consider a short exact sequence:

$$(\dagger) \qquad 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

We say that $(\dagger)$ *splits* if there is a homomorphism $\nu : C \to B$ such that $\psi\nu$ is the identity map on C.

Show that the following conditions are equivalent:

(a) $(\dagger)$ splits;

(b) there is a homomorphism $\mu : B \to A$ such that $\mu\phi$ is the identity map on A.

(c) $B \cong A \oplus C$.

(d) The map $Hom_R(C, B) \to Hom_R(C, C)$ induced by composing with ψ is surjective.

(9) Let R be a ring and let I be an ideal and $a \in R$. Make sense of the following maps and show that the sequence is exact.

$$0 \to R/(I : a) \xrightarrow{a} R/I \to R/(I + (a)) \to 0$$

Here the first map is multiplication by a.