Commutative Algebra, Problem Set 11

Definition: Let $P \subset A$ be a prime ideal. The *height* of P is the Krull dimension of the local ring A_P.

(1) Let A be noetherian local ring. Suppose that there exists a principal prime ideal in A of height at least 1. Prove that A is an integral domain.

(2) Let A be a noetherian integral domain. Show that A is a UFD if and only if every height one prime ideal of A is principal.

(3) Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of Krull dimension d. Show that $d \leq dim_k(\mathfrak{m}/\mathfrak{m}^2)$ with equality if and only if $\mathfrak{m}$ is generated by d elements.

(4) Let A be the polynomial ring $k[x_1, x_2, \ldots]$ over a field. Let $m_1, m_2, \ldots$ be an increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i \geq 2$. For $i \geq 1$, let P_i be the prime ideal $(x_{m_i+1}, x_{m_i+2} \ldots, x_{m_{i+1}})$ and let $S = A \setminus (\cup_{i \geq 1} P_i)$.

Show that S is multiplicative. Then show that the ring $S^{-1}A$ is noetherian (use Problem Set 5, Problem 1) but has infinite Krull dimension (consider the ideals $P_i(S^{-1}A)$).

(5) Find a ring A and a non-unit $x \in A$ such that $\dim(A/(x)) < \dim(A) - 1$.

One possible approach is the following: Take $R = k[x, y]$ and consider two prime ideals P_1 and P_2 of height 1 and 2 respectively such that P_1 and P_2 are incomparable. Then take $S = R \setminus (P_1 \cup P_2)$ and $A = S^{-1}R$. Argue that the maximal ideals of A are P_1A and P_2A. Conclude that A has dimension 2. Finally find an element x in A such that $A/(x)$ is a field.

(6) Let k be a field and let $A = k[X, Y]$ be the polynomial ring in two variables. Consider A as a graded ring with $A_0 = k$, $\deg(X) = 1$ and $\deg(Y) = 2$.

Let $h(n) = dim_k(A_n)$ be the Hilbert function of A. Show that $h(n) = \lfloor \frac{n}{2} \rfloor + 1$ and that this function is not represented by a polynomial for $n >> 0$. Find the Poincare series of A with respect to h.

This shows that the Hilbert function is not always represented by a polynomial.

(7) In this problem we work with graded rings A over a field in the usual grading (that is: variables have degree 1). Consider the two functions:

$h(n) = dim_k(A_n)$ and $f(n) = dim_k(A_0) + dim_k(A_1) + \ldots + dim_k(A_n)$.

Compute these functions and find the polynomials which represent them.

(a) $A = k[X]/(X^2)$,
(b) $A = k[X, Y]/(X^2)$,
(c) $A = k[X, Y]/(X^2, XY)$,
(d) $A = k[X, Y, Z]/(X + Y)$.

Compare the degrees of these polynomials with the dimension of the ring in question.

(8) Let $R = k[X, Y]_{(X,Y)}$ and $A = R/(xy^2)$. Let $I = (x^2, y) \subset A$. Find the polynomial which represents the function $\lambda(A/I^n)$ for $n >> 0$.

(9) Let $A = k[X, Y]_{(X,Y)}$ and let $\mathfrak{m} = (x, y)$ be the maximal ideal. Compute the Hilbert function $\lambda(A/\mathfrak{m}^n)$ and the corresponding Hilbert polynomial.

(10) Let $R = k[X, Y, Z]_{(X,Y,Z)}$. Let $f \in R$ be a homogeneous polynomial of degree d and monic in x (monic in x means that if f is a treated as a polynomial in x, then the leading coefficient is a unit).

Let $A = R/(f)$. Denote by $\mathfrak{m}$ the maximal ideal of the local ring A. Let $I = (y, z)$ be an ideal of A. Show that I is $\mathfrak{m}$-primary. Compute Hilbert function $\lambda(A/I^n)$ and the corresponding polynomial .