Commutative Algebra, Problem Set 10

(1) Show that any finitely generated ideal in a valuation ring is principal.
(2) Let A be a DVR with quotient field k. Show that A is a maximal propert subring of k.
(3) Show that a valuation ring which is not a field, is noetherian if and only if it is a DVR.
(4) Let $(A, \mathfrak{m})$ be a local domain which is not a field such that $\mathfrak{m}$ is principal and $\cap_{n=1}^{\infty} \mathfrak{m}^n = 0$. Show that A is a DVR.
(5) Let A be a Dedekind domain and let S be a multiplicative set in A. Show that $S^{-1}A$ is either a Dedekind domain or the quotient field of A.
(6) Let k be a field and let $A = k[X, Y]$. Let $f \in A$ be an irreducible polynomial such that $f(0, 0) = 0$. Write $f = l + g$ where l is the linear part of f. In other words, $l = aX + bY$ for some $a, b \in k$ and $g \in (X, Y)^2$. Let $R = A/(f)$ and $P = (X, Y)R$.

    Prove that R_P is a DVR if and only of $l \neq 0$.
(7) Let A be a Dedekind domain and let $f = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial over A. Define the *content* of f to be the ideal $c(f) = (a_0, \ldots, a_n)$. Prove the *Gauss lemma*: $c(fg) = c(f)c(g)$ for $f, g \in A[x]$.
(8) Let A be a Dedekind domain and let I be a nonzero ideal. Show that every ideal in A/I is principal. Conclude that every ideal of A can be generated by at most two elements. (Hint: Use Chinese reminder theorem)
(9) Show that a Dedekind domain is a UFD if and only if it is a PID.
(10) Let A be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{10}]$. Show that A is a Dedekind domain, but not a PID.
(11) Let k be a field. Let $B = k[X_1, X_2, \ldots]$ and $A = k[X_n^n, X_n^{n+1} | n \geq 1] \subset B$. Let S be the subset of A consisting of polynomials that, as elements of B, do not have any variable as a factor.Then S is multiplicative. Set $R = S^{-1}A$.

    Verify that R is a noetherian ring of dimension 1 by proving that the only primes of R are 0 and $P_n = (X_n^n, X_n^{n+1})R$ for every n.

    Show that the integral closure of R is $S^{-1}B$. Finally prove that $S^{-1}B$ is not finitely generated as an R-module. Note that the localized module $(S^{-1}B)_{R \setminus P_n}$ is minimally generated by n elements over R_{P_n}.