Commutative Algebra, Problem Set 1

(1) Let R be a ring, and I an ideal consisting of nilpotent elements. Suppose that $a \in R$ maps to a unit in R/I. Then show that a is a unit of R.
(2) Show that if R is a UFD then $R[X]$ is also a UFD.
(3) Show that prime elements in a ring are irreducible. Show that irreducible elements in a UFD are prime. Is this true without the UFD assumption?
(4) Let R be an integral domain. Show that the following are equivalent.
 (a) R is a UFD.
 (b) Every irreducible element of R is prime and principal ideals satisfy the ascending chain condition.
(5) (a) Is intersection of prime ideals a prime ideal?
(b) Suppose that $\{P_\alpha\}$ is a family of prime ideals and assume that this family is totally ordered by inclusion. Then show that $\cap_\alpha P_\alpha$ is also prime.
(6) **(Prime avoidance)**
 (a) Suppose that I is an ideal of a ring R and $P_1, \ldots, P_n$ are prime ideals of R. If $I \not\subset P_i$ for all i, then $I \not\subset \cup_i P_i$.
 (b) Consider the ideal $I = (x, y)$ in the polynomial ring $k[x, y]$, where k is a field. Show that I is contained in an infinite union of primes P_i such that no P_i contains I.
(7) Let I be a finitely generated ideal of a ring R satisfying $I^2 = I$. Then show that I is generated by an idempotent.
(8) Show that an ideal maximal among the family of nonprincipal ideals is prime.
(9) Let R be a ring and let $P \subset R$ be a prime ideal. Show that P contains a minimal prime ideal (that is: there is a minimal element in the class of prime ideals contained in P).
(10) **(Chinese Remainder Theorem)** Let R be a ring and let I, J be two ideals of R. Suppose that I and J are *coprime*, that is: $I + J = R$. Then show that
 (a) $IJ = I \cap J$
 (b) $R/IJ \cong R/I \times R/J$.
(11) Let R be a ring and let J be its Jacobson ideal. Given $a \in R$, show that $a \in J$ if and only if $1 + ab$ is a unit for all $b \in R$.
(12) Find the nilpotent elements of the ring $\mathbb{Z}/n\mathbb{Z}$ for a positive integer $n \geq 1$.
(13) Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module. Is $\mathbb{Q}$ finitely generated as a $\mathbb{Z}$-module?
(14) Let R be a ring containing a field k. Assume that R is a finite dimensional vector space over k. Show that R is artinian and noetherian.
(15) Let R be an artinian domain. Show that R is a field.