ON THE SESHADRI CONSTANTS OF EQUIVARIANT BUNDLES OVER BOTT-SAMELSON VARIETIES AND WONDERFUL COMPACTIFICATIONS

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Abstract. We study torus-equivariant vector bundles E on a complex projective variety X which is either a Bott-Samelson-Demazure-Hansen variety or a wonderful compactification of a complex symmetric variety of minimal rank. We show that E is nef (respectively, ample) if and only if its restriction to every torus–invariant curve in X is nef (respectively, ample). We also compute the Seshadri constants $\varepsilon(E, x)$, where $x \in X$ is any point fixed by the action of a maximal torus.

1. introduction

Let X be a complex projective variety. A vector bundle E on X is said to be nef (respectively, ample) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef (respectively, ample) line bundle on the projective bundle $\mathbb{P}(E)$ over X. If E is an ample vector bundle then it follows easily that the restriction of E to any curve in X is also ample. But the converse is not true in general, even when E has rank one and X has dimension two as shown by an example of Mumford (cf. [17, Example 10.6] or [21, Example 1.5.2]).

Nevertheless it is natural to ask the following question in specific situations: if the restriction of E to every curve in X is ample, then under what conditions is E itself ample?

This question has been studied in several situations. An affirmative answer is given when E is a torus-equivariant vector bundle either on a toric variety [18] or on a generalized flag variety [2]. An affirmative answer is also given when E is any line bundle on a flag variety over a projective curve defined over the algebraic closure of a finite field [4]. A related question was studied in [3] for equivariant vector bundles on wonderful compactifications.

In this paper, we study the question and are able to give an affirmative answer in two different cases: Bott-Samelson-Demazure-Hansen (BSDH) varieties and wonderful compactification of symmetric varieties of minimal rank.

Seshadri constants for line bundles on projective varieties were introduced by Demailly in [11]. Let X be a projective variety, and let L be a nef line bundle on X. For a point

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$x \in X$, the *Seshadri constant* of L at $x \in X$ is defined as:

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves C on X passing through x.

Seshadri constants are useful in studying positivity questions and they are a focus of a lot of current research. In recent years, there has been significant work on computing Seshadri constants for line bundles on projective varieties, especially on surfaces. See [1] for a detailed survey on these works.

While most of the works on Seshadri constants have considered the case of line bundles, there has been recent interest in Seshadri constants for vector bundles of arbitrary rank. An explicit definition of these was first given by Hacon [14]; see Definition 3.4 below. For a comprehensive survey and a generalization to the relative setting, see [13].

In this paper, we compute the Seshadri constants of equivariant vector bundles at torus-fixed points in the two cases that we study.

In Section 2, we recall basic definitions, constructions and prove some preparatory lemmas. In Section 3, we prove our main results about positivity of equivariant vector bundles and their Seshadri constants. We show that a torus-equivariant vector bundle E on a BSDH variety or on a wonderful compactification of a symmetric variety of minimal rank is ample if and only its restriction $E|_C$ to every torus-invariant curve C is ample (see Theorem 3.1 and Theorem 3.8). We also compute the Seshadri constants of such vector bundles at torus-fixed points (Theorem 3.5 and Theorem 3.10).

Each section is divided into two subsections dealing with the two cases that we study.

We work throughout over the field $\mathbb{C}$ of complex numbers. The field of real numbers is denoted by $\mathbb{R}$.

2. PRELIMINARIES

In this section, we recall basic definitions and prove some preparatory results.

2.1. **BSDH varieties.** Let G be a semisimple and simply connected algebraic group of adjoint type defined over $\mathbb{C}$. We fix a maximal torus T of G; let $W = N_G(T)/T$ be the Weyl group of G with respect to T. We denote by R the set of roots of G with respect to T and by R^+ a set of positive roots. Let B^+ be the Borel subgroup of G containing T with respect to R^+. Let w_0 denote the longest element of the Weyl group W. Let B be the Borel subgroup of G opposite to B^+ determined by T, i.e., $B = n_{w_0} B^+ n_{w_0}^{-1}$, where n_{w_0} is a representative of w_0 in $N_G(T)$. Note that the set of roots of B coincides with the set $R^- := -R^+$ of negative roots. Let

$$S = \{\alpha_1, \cdots, \alpha_n\}$$

be the set of all simple roots in R^+, where n is the rank of G. The simple reflection in the Weyl group corresponding to a simple root α is denoted by s_α. For simplicity of notation, the simple reflection corresponding to a simple root α_i is denoted by s_i.

By $\mathfrak{g}$ (respectively, $\mathfrak{t}$) we denote the Lie algebra of G (respectively, T); similarly, the Lie algebra of B is denoted by $\mathfrak{b}$. Let

$$X(T) \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$$

be the dual of the real form $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{t}$. The positive definite $W(G, T)$–invariant form on $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form on $\mathfrak{g}$ is denoted by $(-, -)$. We define

$$\langle \nu, \alpha \rangle := \frac{2(\nu, \alpha)}{(\alpha, \alpha)}, \text{ for } \alpha \in R.$$

In this set-up, there is the Chevalley basis

$$\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in S\}$$

of $\mathfrak{g}$ determined by T. For a root α, we denote by U_α (respectively, $\mathfrak{g}_\alpha$) the one-dimensional T-stable root subgroup of G (respectively, subspace of $\mathfrak{g}$) on which T acts through the character α.

For any $w \in W$, let

$$X(w) := \overline{BwB/B} \subset G/B$$

be the corresponding Schubert variety in G/B. Given a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ of w, the Bott-Samelson-Demazure-Hansen variety

$$Z(w, \underline{i}) \tag{2.1}$$

is a desingularization of the Schubert variety $X(w)$, where $\underline{i} := (i_1, \cdots, i_r)$. For a simple root $\alpha \in S$, let $n_\alpha \in N_G(T)$ be a representative of s_α. The minimal parabolic subgroup of G containing B and n_α is denoted P_α. Then $Z(w, \underline{i})$ in (2.1) is defined as

$$Z(w, \underline{i}) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}}{B \times B \times \cdots \times B},$$

where the action of $B \times B \times \cdots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}}$ is given by

$$(p_1, p_2, \cdots, p_r)(b_1, b_2, \cdots, b_r) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \cdots, b_{r-1}^{-1} \cdot p_r \cdot b_r)$$

for $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$ and $\underline{i} = (i_1, i_2, \cdots, i_r)$ (see [10, Definition 1, p.73], [8, Definition 2.2.1, p.64]). We recall that $Z(w, \underline{i})$ is a smooth projective variety. Let

$$\phi_w : Z(w, \underline{i}) \longrightarrow X(w) \tag{2.2}$$

be the natural birational surjective morphism.

For the sake of simplicity, we will refer to any Bott-Samelson-Demazure-Hansen variety simply as a BSDH variety. For more details on BSDH varieties, see [5, 10, 15].

Let

$$f_r : Z(w, \underline{i}) \longrightarrow Z(ws_{i_r}, \underline{i'}) \tag{2.3}$$

be the morphism induced by the projection

$$P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_r}} \longrightarrow P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \cdots \times P_{\alpha_{i_{r-1}}},$$

where $\underline{i'} = (i_1, i_2, \cdots, i_{r-1})$. The map f_r in (2.3) is in fact a $P_{\alpha_{i_r}}/B \simeq \mathbb{P}^1$ fibration. There is a natural B-equivariant imbedding

$$Z(w, \underline{i}) \hookrightarrow G/P_{S\setminus\{\alpha_{i_1}\}} \times G/P_{S\setminus\{\alpha_{i_2}\}} \times \cdots \times G/P_{S\setminus\{\alpha_{i_r}\}}\,; \tag{2.4}$$

see [23, Theorem 1(ii), Page 608].

For any $\beta \in R^+$, let $C_\beta := \overline{U_\beta s_\beta B/B} \subset G/B$. For any $v \in W^{S\setminus\{\alpha_{i_j}\}}$ and $\beta \in R^+(v^{-1})$, let

$$C_{\beta,v,j} := \overline{U_\beta v P_{S\setminus\{\alpha_{i_j}\}}/P_{S\setminus\{\alpha_{i_j}\}}} \subset G/P_{S\setminus\{\alpha_{i_j}\}}\,.$$

Note that C_β is isomorphic to the projective line $\mathbb{P}^1$ for every β. Indeed, the orbit $U_\beta s_\beta B/B$ is isomorphic to the affine line $\mathbb{A}^1$. The complement $C_\beta \setminus (U_\beta s_\beta B/B)$ consists of a single T-fixed point. Consequently, C_β is isomorphic to $\mathbb{P}^1$.

Lemma 2.1. *Let $v \in W^{S\setminus\{\alpha_{i_j}\}}$. For any $\beta \in R^+(v^{-1})$, there is a unique T-equivariant isomorphism $\sigma_{\beta,v,j} : C_\beta \longrightarrow C_{\beta,v,j}$ such that $\sigma_{\beta,v,j}(gs_\beta B/B) = gvP_{S\setminus\{\alpha_{i_j}\}}/P_{S\setminus\{\alpha_{i_j}\}}$ for all $g \in U_\beta$.*

Proof. Clearly, there is an isomorphism $\tau_{\beta,v,j} : U_\beta s_\beta B/B \longrightarrow U_\beta v P_{S\setminus\{\alpha_{i_j}\}}$ such that $\tau_{\beta,v,j}(gs_\beta B/B) = gvP_{S\setminus\{\alpha_{i_j}\}}/P_{S\setminus\{\alpha_{i_j}\}}$ for all $g \in U_\beta$. Now, $\tau_{\beta,v,j}$ can be extended uniquely to entire C_β by the valuative criterion. □

Let

$$Y := Y(w, \underline{i}) := G/P_{S\setminus\{\alpha_{i_1}\}} \times G/P_{S\setminus\{\alpha_{i_2}\}} \times \cdots \times G/P_{S\setminus\{\alpha_{i_r}\}}. \tag{2.5}$$

Lemma 2.2. *Let $\beta \in R^+$. Let $(v_1, v_2, \cdots, v_r) \in W^{S\setminus\{\alpha_{i_1}\}} \times W^{S\setminus\{\alpha_{i_1}\}} \times \cdots \times W^{S\setminus\{\alpha_{i_r}\}}$ be such that there is an integer $1 \le j \le r$ for which $v_j^{-1}(\beta)$ is negative. Let $v := (v_1, v_2, \cdots, v_r)$. Let $A \subset \{1, 2, \cdots, r\}$ be a non-empty subset such that $\beta \in \bigcap_{j\in A} R^+(v_j^{-1})$. Then there is a unique T-equivariant morphism*

$$\phi_{\beta,v,A} : C_\beta \longrightarrow Y$$

(see (2.5)) satisfying the following conditions:

- *$\pi_j \circ \phi_{\beta,v,A} = \sigma_{\beta,v_j,j}$ for every $j \in A$, and*
- *$\pi_j \circ \phi_{\beta,v,A} = v_j P_{S\setminus\{\alpha_{i_j}\}}/P_{S\setminus\{\alpha_{i_j}\}}$ for every $j \in \{1, 2, \cdots, r\} \setminus A$.*

Proof. For $x \in C_\beta$, define $\phi_{\beta,v,A}(x) = (x_1, x_2, \cdots, x_r)$, where $x_j = \sigma_{\beta,v_j,j}(x)$ if $j \in A$, and $x_j = v_j P_{S\setminus\{\alpha_{i_j}\}}/P_{S\setminus\{\alpha_{i_j}\}}$ if $j \in \{1, 2, \cdots, r\} \setminus A$. □

Lemma 2.3. *For any T–invariant curve C in Y (see (2.5)), there exist the following:*

1. *$v_j \in W^{S\setminus\{\alpha_{i_j}\}}$ and $t_j \in T$ for every $1 \le j \le r$,*
2. *a non-empty subset $A \subset \{1, 2, \cdots, r\}$, and*
3. *a root $\beta \in \bigcap_{j\in A} R^+(v_j^{-1})$,*

such that

$$C = (t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(C_\beta),$$

where $v = (v_1, v_2, \cdots, v_r)$.

Proof. Let $\pi_j : Y \longrightarrow G/P_{S\setminus\{\alpha_{i_j}\}}$ be the projection to the j-th factor in (2.5). Let $z_k = P_{S\setminus\{\alpha_{i_k}\}}/P_{S\setminus\{\alpha_{i_k}\}}$ for every $1 \leq k \leq r$. Let

$$A := \{j \in \{1, 2, \cdots, r\} \mid \pi_j(C) \text{ is a curve}\}.$$

So, for any $j \in \{1, 2, \cdots, r\} \setminus A$, the image $\pi_j(C)$ is a T-fixed point, and hence it is of the form $v_j z_j$ for some $v_j \in W^{S\setminus\{\alpha_{i_j}\}}$. Further, for every $j \in A$, there is an element $v_j \in W^{S\setminus\{\alpha_{i_j}\}}$ and a root $\beta_j \in R^+(v_j^{-1})$ such that $\pi_j(C) = \sigma_{\beta_j, v_j, j}(C_{\beta_j})$. Since C is a curve, we have $\beta_j = \beta_k$ for every pair of distinct elements j and k of A.

Let $\beta := \beta_j$ for some $j \in A$, and take $x_0 = u_\beta(1)s_\beta B/B$. Now, let $y_0 \in C$ be a point such that $\pi_j(y_0)$ is not fixed by T for any $j \in A$. Then for any $j \in A$, there is an element $t_j \in T$ for which $\pi_j(y_0) = t_j\sigma_{\beta,v_j,j}(x_0)$. Take $t_j = 1$ for all $j \in \{1, 2, \cdots, r\} \setminus A$. Let $v = (v_1, v_2, \cdots, v_r)$. Therefore, $(t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(x_0) \in C$ and it is not fixed by T. The condition that C is T–invariant implies that the T-orbit of $T{\cdot}(t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(x_0)$ is an open dense subset of C. Since $(t_1, t_2, \cdots, t_r)$ commutes with every element of T (this is because T is the diagonal subgroup of T^r), we conclude that

$$T \cdot (t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(x_0) = (t_1, t_2, \ldots, t_r).T \cdot \phi_{\beta,v,A}(x_0)$$

is an open dense subset of $(t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(C_\beta)$. Thus, we have

$$C = (t_1, t_2, \cdots, t_r).\phi_{\beta,v,A}(C_\beta),$$

and the proof is complete. □

2.2. **Wonderful compactifications.** Let G be a semisimple adjoint type algebraic group over $\mathbb{C}$. We use the notation introduced at the beginning of Section 2.1.

Let σ be an automorphism of G of order two. Let $H = G^\sigma \subset G$ be the subgroup defined by the fixed point locus of σ. We now recall some properties of the symmetric variety G/H.

A torus T' of G is said to be σ-*anisotropic* if $\sigma(t) = t^{-1}$ for every $t \in T'$. The rank of G/H is the dimension of a maximal dimensional σ-anisotropic torus.

If T is a σ–invariant maximal torus of G, then σ induces an automorphism of $X(T)$ of order two. Evidently, $\sigma(R) = R$. Further, we have

$$T = T_1 T_2, \tag{2.6}$$

where T_1 and T_2 are tori with the property that $\sigma(t) = t$ for every $t \in T_1$ and $\sigma(t) = t^{-1}$ for every $t \in T_2$. Clearly, $T_1 \bigcap T_2$ is a finite group, and $\text{rank}(G/H) \geq \text{rank}(G) - \text{rank}(H)$.

Throughout, we assume that G/H is of *minimal rank*, meaning

$$\text{rank}(G/H) = \text{rank}(G) - \text{rank}(H).$$

See [6, 19, 24, 25] for more details.

Let

$$X := \overline{G/H} \tag{2.7}$$

be the wonderful compactification of G/H, constructed in [9]. Let T be a σ-stable maximal torus of G for which $\dim T_1$ (see (2.6)) is maximal, and let B be a Borel subgroup of G

containing T satisfying the condition that for any root $\alpha \in R^+(B)$ either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) \in -R^+(B)$.

We will use the following result which is a consequence of [7, p. 482, Lemma 2.1.1].

Lemma 2.4. *Let z be the unique B-fixed point of X in (2.7). For a positive root $\alpha \in R^+(B) \setminus R^+(L)$, let $C_{z,\alpha} = \overline{U_\alpha s_\alpha z}$. For a restricted root $\gamma = \alpha - \sigma(\alpha)$, let $C_{z,\gamma}$ be the unique T–invariant curve containing z on which T acts through the character γ. Then the irreducible T–invariant curves in X are the W-translates of the curves $C_{z,\alpha}$ and $C_{z,\gamma}$. They are all isomorphic to $\mathbb{P}^1$.*

3. Equivariant vector bundles

3.1. **Equivariant bundles on BSDH varieties.** In this subsection we investigate positivity of torus–equivariant vector bundles on BSDH varieties. We follow the notation of Section 2.1.

Theorem 3.1. *Let G be a semisimple and simply connected algebraic group over the complex numbers. Let T be a maximal torus of G, and let B be a Borel subgroup of G containing T as in Section 2.1. Let $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ be a reduced expression and $\underline{i} := (i_1, \cdots, i_r)$. Let $Z(w, \underline{i})$ be the corresponding Bott-Samelson-Demazure-Hansen variety.*

Let E be a T–equivariant vector bundle on $Z(w, \underline{i})$. Then E is nef (respectively, ample) if and only if the restriction $E\big|_C$ of E to every T–invariant curve C on $Z(w, \underline{i})$ is nef (respectively, ample).

Proof. Let $Y(T)$ be the group of all 1–parameter subgroups of T. Note that $Y(T)$ is a finitely generated free abelian group whose rank coincides with the dimension of T. Let

$$\{\lambda_1, \cdots, \lambda_n\} \tag{3.1}$$

be a basis of the $\mathbb{Z}$–module $Y(T)$.

If E is nef (respectively, ample), then clearly $E\big|_C$ is nef (respectively, ample) for every curve C on $Z(w, \underline{i})$.

To prove the converse, first assume that E is a T–equivariant vector bundle on $Z(w, \underline{i})$ such that its restriction $E\big|_C$ to every T–invariant curve $C \subset Z(w, \underline{i})$ is nef.

Let

$$p : \mathbb{P}(E) \longrightarrow Z(w, \underline{i})$$

be the projective bundle over $Z(w, \underline{i})$ associated to E. Let $\mathcal{O}_{\mathbb{P}(E)}(1)$ denote the tautological relative ample line bundle over $\mathbb{P}(E)$. By definition, E is nef if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is nef. So to prove that E is nef, it suffices to show that $\mathcal{O}_{\mathbb{P}(E)}(1)\big|_D$ is nef for every curve $D \subset \mathbb{P}(E)$. If $p(D)$ is a point, then $\mathcal{O}_{\mathbb{P}(E)}(1)\big|_D$ is ample, because $\mathcal{O}_{\mathbb{P}(E)}(1)$ is relatively ample.

Suppose now that $p(D)$ is a curve in $Z(w, \underline{i})$. Let $\widetilde{D_1}$ be the flat limit of the curves $\lambda_1(t)D$ (see (3.1)) as $t \to 0$. In other words, $\widetilde{D_1}$ is a 1–cycle which corresponds to the limit of the points $\lambda_1(t)[D]$, as $t \to 0$, in the Hilbert scheme of curves in $\mathbb{P}(E)$. Note that since E is T–equivariant, the 1–parameter subgroup λ_1 acts on the Hilbert scheme of curves in $\mathbb{P}(E)$. It follows that the 1–cycle $\widetilde{D_1}$ and $D_1 := p(\widetilde{D_1})$ are in fact λ_1–invariant. Now let $\widetilde{D_2}$ be the flat limit of $\lambda_2(t)\widetilde{D_1}$ as $t \to 0$. Since λ_1 and λ_2 commute, we see that $\widetilde{D_2}$ and $p(\widetilde{D_2})$ are invariant under both λ_1 and λ_2. Continuing this way, we obtain a 1–cycle $\widetilde{D_n}$ on $\mathbb{P}(E)$ such that both $\widetilde{D_n}$ and $p(\widetilde{D_n})$ are invariant under $\lambda_1, \cdots, \lambda_n$. Since $\{\lambda_1, \cdots, \lambda_n\}$ is a basis of $Y(T)$, both $\widetilde{D_n}$ and $p(\widetilde{D_n})$ are in fact T–invariant.

Now, by the assumption on E, we have

$$\text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)\big|_{\widetilde{D_n}}) \geq 0 . \tag{3.2}$$

Since the curve D is linearly equivalent to $\widetilde{D_n}$, from (3.2) it follows immediately that $\text{degree}(\mathcal{O}_{\mathbb{P}(E)}(1)\big|_D) \geq 0$. This proves that E is nef if $E\big|_C$ is nef for every T–invariant curve $C \subset Z(w, \underline{i})$.

Next suppose that $E\big|_C$ is ample for every T–invariant curve $C \subset Z(w, \underline{i})$. We claim that E is ample.

Recall from (2.4) and (2.5) that we have an embedding

$$Z(w, \underline{i}) \hookrightarrow Y(w, \underline{i}) = G/P_{S\setminus\{\alpha_{i_1}\}} \times G/P_{S\setminus\{\alpha_{i_2}\}} \times \cdots \times G/P_{S\setminus\{\alpha_{i_r}\}}.$$

It can be shown inductively that the operation of restriction of line bundles gives an isomorphism of the Picard groups

$$\text{res} : \text{Pic}(Y(w, \underline{i})) \longrightarrow \text{Pic}(Z(w, \underline{i})); \tag{3.3}$$

see [20, Section 3.1, Page 464] for more details.

Further, every line bundle on $Y(w, \underline{i})$ is T^r-equivariant. In particular, every line bundle on $Z(w, \underline{i})$ is T–equivariant, for the action defined using the diagonal inclusion of T in T^r. Hence if F is a T–equivariant vector bundle on $Z(w, \underline{i})$, then so is $F \otimes L$ for any line bundle L on $Z(w, \underline{i})$.

To prove that E is ample, fix an ample line bundle L on $Z(w, \underline{i})$. Let $\text{Sym}^n(E)$ denote the n–th symmetric power of E.

In view of the isomorphism res in (3.3), we may assume that L is actually a line bundle on $Y(w, \underline{i})$. For any T–invariant curve C in $Y(w, \underline{i})$, let a_C denote the positive integer such that $L\big|_C = \mathcal{O}_C(a_C)$; recall that $C \cong \mathbb{P}^1$ and $L\big|_C$ is ample. Then for $t \in T^r$, we have $a_C = a_{t\cdot C}$ for every T–invariant curve C in $Y(w, \underline{i})$.

By Lemma 2.3, up to T^r-translates, there are only finitely many T–invariant curves in $Y(w, \underline{i})$. Therefore the collection of integers a_C as C varies over T–invariant curves in $Y(w, \underline{i})$ is finite. In particular, this collection of integers a_C, as C varies over T–invariant curves in $Z(w, \underline{i})$, is also finite.

Take a T–invariant curve $C \subset Z(w, \underline{i})$. Since $E\big|_C$ is ample, the vector bundle $\text{Sym}^n(E)\big|_C$ is also ample for every integer $n \geq 1$. Moreover, $\text{Sym}^n(E)\big|_C$ is a direct sum of line bundles and the degrees of these line bundles increase with n. Since the collection of integers a_C such that $L\big|_C = \mathcal{O}_C(a_C)$ is finite as C varies over the T–invariant curves in $Z(w, \underline{i})$, we can choose a sufficiently large integer n such that $\text{Sym}^n(E) \otimes L^{-1}\big|_C$ is nef for every T–invariant curve C. By the first part of the theorem, this vector bundle $\text{Sym}^n(E) \otimes L^{-1}$

is nef. Since L is ample, this implies that $\mathrm{Sym}^n(E) = (\mathrm{Sym}^n(E) \otimes L^{-1}) \otimes L$ is ample, and consequently E itself is ample (see [22, Proposition 6.2.11] and [16, p. 67, Proposition 2.4]). □

Remark 3.2. A similar result was proved for equivariant vector bundles on toric varieties in [18, Theorem 2.1] and on generalized flag varieties in [2, Theorem 3.1]. Our proof of Theorem 3.1 above is motivated by these results.

Proposition 3.3. *Let G, T, B, $Z(w, \underline{i})$ and $Y(w, \underline{i})$ be as in Theorem 3.1. Let $x \in Z(w, \underline{i})$ be a T–fixed point, and let $\pi : \widetilde{X} \longrightarrow Z(w, \underline{i})$ denote the blow-up of $Z(w, \underline{i})$ at x. Then the following three statements hold:*

(1) *The action of T on $Z(w, \underline{i})$ lifts to $\widetilde{X}$.*
(2) *Let F be a T–equivariant vector bundle on $\widetilde{X}$. Then F is nef if and only if the restriction $F\big|_{\widetilde{C}}$ of F to every T–invariant curve $\widetilde{C} \subset \widetilde{X}$ is nef.*
(3) *Let W_x denote the exceptional divisor of the above blow-up π. Let E be a T–equivariant vector bundle on $Z(w, \underline{i})$. Then $(\pi^\star E) \otimes \mathcal{O}_{\widetilde{X}}(m \cdot W_x)$ is a T–equivariant vector bundle on $\widetilde{X}$ for every integer m.*

Proof. Let $x \in Z(w, \underline{i})$ be a T–fixed point. The tangent space $T_x(Y(w, \underline{i}))$ is isomorphic to the T-module

$$\bigoplus_{j=1}^{r} \mathfrak{g}/\mathfrak{p}_{S \setminus \{\alpha_{i_j}\}},$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{p}_{S\setminus\{\alpha_{i_j}\}}$ is the Lie algebra of $P_{S\setminus\{\alpha_{i_j}\}}$. Hence $T_x(Z(w, \underline{i}))$ is a T–invariant subspace of $\bigoplus_{j=1}^{r} \mathfrak{g}/\mathfrak{p}_{S\setminus\{\alpha_{i_j}\}}$. So the action of T on $T_x(Z(w, \underline{i}))$ is linear.

Note that the exceptional divisor of the blow-up $\pi : \widetilde{X} \longrightarrow Z(w, \underline{i})$ is isomorphic to $\mathbb{P}(T_x(Z(w, \underline{i})))$. So T acts on the exceptional divisor of the blow-up via the linear action of T on $T_x(Z(w, \underline{i}))$. Since π is an isomorphism outside the exceptional divisor, we conclude that the action of T on $Z(w, \underline{i})$ lifts to entire $\widetilde{X}$. This proves (1).

Since the action of T lifts to $\widetilde{X}$, the proof of (2) goes through exactly as the proof of the analogous statement in Theorem 3.1 does. Note that in the proof of the nef part of Theorem 3.1 we only used the T–action on $Z(w, \underline{i})$.

Now we will prove (3). Since T acts on the exceptional divisor W_x of π, we conclude that $\mathcal{O}_{\widetilde{X}}(m \cdot W_x)$ is a T–equivariant line bundle for every integer m. Hence if E is a T–equivariant vector bundle on $Z(w, \underline{i})$, then $(\pi^\star E) \otimes \mathcal{O}_{\widetilde{X}}(m \cdot W_x)$ is a T–equivariant vector bundle on $\widetilde{X}$ for every integer m. □

We now recall the definition of Seshadri constant for vector bundles. Let E be a vector bundle on a projective variety X. Take any point $x \in X$. The Seshadri constant of E at x was first defined explicitly in [14].

Let $\pi : \widetilde{X} \longrightarrow X$ denote the blow-up of X at x. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}(\pi^\star E) & \xrightarrow{\widetilde{\pi}} & \mathbb{P}(E) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \widetilde{X} & \xrightarrow{\pi} & X \end{array} \tag{3.4}$$

so the above projections p and q define projective bundles. Let

$$\xi = \mathcal{O}_{\mathbb{P}(\pi^\star E)}(1) \tag{3.5}$$

be the tautological bundle on $\mathbb{P}(\pi^\star E)$. Let $Y_x = p^{-1}(x)$ and $Z_x = \widetilde{\pi}^{-1}(Y_x)$.

We can now give the definition of Seshadri constants of vector bundles.

Definition 3.4. *Let X be a projective variety and let E be a vector bundle on X. For a point $x \in X$, the Seshadri constant of E at x is defined as*

$$\varepsilon(E, x) := \sup\{\lambda \geq 0 \mid \xi - \lambda Z_x \text{ is nef}\}.$$

Alternately, $\varepsilon(E, x)$ can be defined as follows:

$$\varepsilon(E, x) = \inf \frac{\xi \cdot C}{\mathrm{mult}_x p_* C},$$

where the infimum is taken over all curves $C \subset \mathbb{P}(E)$ that intersect $p^{-1}(x)$, but are not contained in $p^{-1}(x)$. For more details on Seshadri constants of vector bundles, see [14, 13].

Now we work with the notation in Theorem 3.1. Let E be a T–equivariant vector bundle on $Z(w, \underline{i})$. Let $C \subset Z(w, \underline{i})$ be a T–invariant curve. Since $C \cong \mathbb{P}^1$, the restriction of E to C is of the form

$$E\big|_C = \mathcal{O}_C(a_1) \oplus \ldots \oplus \mathcal{O}_C(a_n)$$

for some integers $a_1, \cdots, a_n$.

The next result uses this decomposition to describe the Seshadri constants of E at T-fixed points.

Theorem 3.5. *Let G be a semisimple and simply connected algebraic group. Let T be a maximal torus of G, and let B be a Borel subgroup of G containing T as in Theorem 3.1. Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression, and $\underline{i} := (i_1, \cdots, i_r)$. Let $X := Z(w, \underline{i})$ be the corresponding Bott-Samelson-Demazure-Hansen variety. Let E be a T–equivariant nef vector bundle on X of rank n, and let $x \in X$ be a T–fixed point. Then*

$$\varepsilon(E, x) = \min\{a_i(C)\}_{C,i},$$

where the minimum is taken over all T–invariant curves $C \subset X$ passing through x and integers $\{a_1(C), \cdots, a_n(C)\}$ such that $E\big|_C = \mathcal{O}_C(a_1(C)) \oplus \cdots \oplus \mathcal{O}_C(a_n(C))$.

Proof. Let W_x denote the exceptional divisor of the blow-up

$$\pi : \widetilde{X} \longrightarrow X = Z(w, \underline{i})$$

at the point $x \in Z(w, \underline{i})$. By definition, the Seshadri constant of E at x is given by the following (see (3.4)):

$$\varepsilon(E, x) = \sup\{\lambda \geq 0 \mid \xi - \lambda q^\star(W_x) \text{ is nef}\}.$$

To prove that $\xi - \lambda q^\star(W_x)$ is nef for a particular λ, we need to show that

$$(\xi - \lambda q^\star(W_x)) \cdot D \geq 0,$$

for every curve $D \subset \mathbb{P}(\pi^\star E)$. As argued in the proof of Theorem 3.1, there exists a T–invariant curve $\widetilde{D} \subset \mathbb{P}(\pi^\star E)$ which is linearly equivalent to D and moreover

$$\widetilde{C} := q(\widetilde{D}) \subset \widetilde{X}$$

is a T–invariant curve.

Consider the following diagram:

$$\begin{array}{ccccc} \mathbb{P}(\pi^\star E\big|_{\widetilde{C}}) & \hookrightarrow & \mathbb{P}(\pi^\star E) & \xrightarrow{\widetilde{\pi}} & \mathbb{P}(E) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \widetilde{C} & \hookrightarrow & \widetilde{X} & \xrightarrow{\pi} & X \end{array} \tag{3.6}$$

Then

$$(\xi - \lambda q^\star(W_x)) \cdot D = (\xi - \lambda q^\star(W_x)) \cdot \widetilde{D} = \left[\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x\big|_{\widetilde{C}})\right] \cdot \widetilde{D}.$$

So $\xi - \lambda q^\star(W_x)$ is nef if and only if $\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x\big|_{\widetilde{C}})$ is nef for every T–invariant curve $\widetilde{C} \subset \widetilde{X}$.

Now let $\widetilde{C} \subset \widetilde{X}$ be any T–invariant curve. We know that $\widetilde{C}$ is isomorphic to the projective line $\mathbb{P}^1$. We will now investigate the values $\lambda \geq 0$ for which the line bundle $\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1) - \lambda q^\star(W_x\big|_{\widetilde{C}})$ is nef.

First suppose that $\widetilde{C}$ is contained in the exceptional divisor W_x of the blow-up

$$\pi : \widetilde{X} \longrightarrow X.$$

Note that W_x is isomorphic to a projective space, and

$$\mathcal{O}_{\widetilde{X}}(W_x)\big|_{W_x} = \mathcal{O}_{W_x}(-1).$$

Hence we have $-\lambda(W_x\big|_{\widetilde{C}}) = \mathcal{O}_{\widetilde{C}}(\lambda)$. Since E on X is nef by hypothesis, it follows that $\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x\big|_{\widetilde{C}})$ is nef for every $\lambda \geq 0$.

Now suppose that $\widetilde{C}$ is not contained in W_x, and set $C = \pi(\widetilde{C})$. Then $C \subset X$ is a T–invariant curve. If $x \notin C$, then $W_x\big|_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}}$, and $\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1)$ is nef because $\pi^\star E\big|_{\widetilde{C}}$ is so.

Finally, assume that $x \in C$. Then

$$W_x \cdot \widetilde{C} = 1,$$

since C is a smooth curve. Let $a_1(C), \cdots, a_n(C)$ be non-negative integers such that $E\big|_C = \mathcal{O}_C(a_1(C)) \oplus \ldots \oplus \mathcal{O}_C(a_n(C))$. Then $\mathcal{O}_{\mathbb{P}(\pi^\star E|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x\big|_{\widetilde{C}})$ is nef if and only if $\mathcal{O}_{\widetilde{C}}(a_1(C) - \lambda) \oplus \cdots \oplus \mathcal{O}_{\widetilde{C}}(a_n(C) - \lambda)$ is nef. Now.

$$\mathcal{O}_{\widetilde{C}}(a_1(C) - \lambda) \oplus \cdots \oplus \mathcal{O}_{\widetilde{C}}(a_n(C) - \lambda)$$

is nef if and only if $\lambda \leq \min\{a_1(C), \cdots, a_n(C)\}$. Running over all T–invariant curves in X, we obtain the theorem. □

Remark 3.6. Similar computations of Seshadri constants were carried out in [18, Proposition 3.2] for torus-equivariant vector bundles on toric varieties, and in [2, Theorem 3.2] for torus-equivariant vector bundles on generalized flag varieties.

Remark 3.7. It is known that a nef vector bundle E on a projective variety X is ample if and only if $\inf_{x \in X} \varepsilon(E, x) > 0$; for example, see [13, Theorem 3.11]. On the other hand, in general, Seshadri constants of ample vector bundles can be arbitrarily small positive numbers (see [21, Example 5.2.1]).

But one expects better lower bounds at very general points. If L is an ample line bundle on X, then [12] proved the following:

$$\varepsilon(L, x) \geq 1, \text{ for a very general point } x \in X.$$

It is interesting to ask if this lower bound can be generalized to vector bundles of arbitrary rank. This question is open in general.

Theorem 3.5 shows that for an ample T–equivariant vector bundle E on $Z(w, \underline{i})$, we have

$$\varepsilon(E, x) \geq 1$$

for every T–fixed point $x \in Z(w, \underline{i})$. If E is a vector bundle on a projective variety X, then the maximum value of $\varepsilon(E, x)$, as the point x runs over X, is achieved at a very general point in X (see [13, Proposition 3.35]). Since T–fixed points are special, we conclude that

$$\varepsilon(E, x) \geq 1$$

for a very general point $x \in Z(w, \underline{i})$ if E is an ample T–equivariant vector bundle on $Z(w, \underline{i})$. This gives an affirmative answer to the above question in the case of ample T–equivariant vector bundles on BSDH varieties.

3.2. **Equivariant bundles on wonderful compactifications.** In this subsection we study positivity of equivariant vector bundles on wonderful compactifications. The notation of Section 2.2 is followed. The proofs are similar to the corresponding proofs in Section 3.1.

Theorem 3.8. *Let G be a semisimple adjoint type algebraic group. Let σ be an automorphism of G of order two, and let H be the subgroup of all fixed points of σ in G. Assume that G/H has minimal rank. Let $X := \overline{G/H}$ be the wonderful compactification of G/H. Let T be a σ-stable maximal torus of G such that* $\dim T_1$ *(see* (2.6)*) is maximal.*

Let E be a T–equivariant vector bundle on X. Then E is nef (respectively, ample) if and only if the restriction $E\big|_C$ of E to every T–invariant curve C on X is nef (respectively, ample).

Proof. The proof for the nef part is the same as the proof of the analogous statement in Theorem 3.1: Suppose that E is a T–equivariant vector bundle on X such that the restriction $E\big|_C$ of E to every T–invariant curve C is nef. If we are given a curve D, then we can construct a T–invariant curve C such that C and D are linearly equivalent. So it follows that E is nef.

Next suppose that $E\big|_C$ is ample for every T–invariant curve $C \subset X$. We will show that E itself is ample. This also is similar to the proof in Theorem 3.1; in fact, it is easier.

Note that every line bundle on X is T-equivariant. Hence if F is a T–equivariant vector bundle on X, then so is $F \otimes L$ for any line bundle L on X. To show that E is ample, fix an ample line bundle L on X. Since every line bundle on X is T-equivariant, $E \otimes L$ is also T-equivariant.

By Lemma 2.4, there are only finitely many T–invariant curves in X. Now we complete the argument exactly as done in the last paragraph of the proof of Theorem 3.1. □

Proposition 3.9. *Let G, X, T and B be as in Theorem 3.8. Let $x \in X$ be a T–fixed point, and let $\pi : \widetilde{X} \longrightarrow X$ denote the blow-up of X at x. Then the following three statements hold:*

1. *The action of T on X lifts to $\widetilde{X}$.*
2. *Let F be a T–equivariant vector bundle on $\widetilde{X}$. Then F is nef if and only if the restriction $F\big|_{\widetilde{C}}$ of F to every T–invariant curve $\widetilde{C} \subset \widetilde{X}$ is nef.*
3. *Let W_x denote the exceptional divisor of the blow-up π. Let E be a T–equivariant vector bundle on X. Then $(\pi^\star E) \otimes \mathcal{O}_{\widetilde{X}}(m \cdot W_x)$ is a T–equivariant vector bundle on $\widetilde{X}$ for every integer m.*

Proof. Let $\lambda \in X(T)$ be a regular special dominant character (see [9, Definition, Page 6]), and let $V_{2\lambda}$ be the rational irreducible G-module with height 2λ. By the construction in [9, Page 10], we have

$$X = \overline{G/H} \subset \mathbb{P}(V_{2\lambda}).$$

Let $x \in X$ be a T–fixed point. Let $G' = \mathrm{GL}(V_\lambda)$, and let P be the stabilizer of x in G'. Then P is the parabolic subgroup of G' containing T. We further have

$$T_x(\mathbb{P}(V_\lambda)) = \mathfrak{gl}(V_\lambda)/\mathfrak{p},$$

where $T_x(\mathbb{P}(V_\lambda))$ denotes the tangent space of $\mathbb{P}(V_\lambda)$ at x and $\mathfrak{p}$ is the Lie algebra of P. Hence $T_x(\mathbb{P}(V_\lambda))$ is a P-module. In particular, it is a T-module. Further, $T_x(X)$ is a T–stable vector subspace of $T_x(\mathbb{P}(V_\lambda))$. So the action of T on $T_x(X)$ is linear.

The rest of the proof is identical to the proof of Proposition 3.3 and we omit it. □

Now we work with the notation in Theorem 3.8. Let E be a T–equivariant vector bundle on X.

Let $C \subset X$ be a T–invariant curve. By Lemma 2.4, C is isomorphic to the projective line $\mathbb{P}^1$. Hence the restriction of E to C is of the form

$$E\big|_C = \mathcal{O}_C(a_1) \oplus \ldots \oplus \mathcal{O}_C(a_n)$$

for some integers $a_1, \cdots, a_n$.

Theorem 3.10. *Let G be a semisimple adjoint type algebraic group. Let σ be an automorphism of G of order two, and let H be the subgroup of all fixed points of σ in G. Assume that G/H has minimal rank. Let $X := \overline{G/H}$ be the wonderful compactification of G/H. Let T be a σ-stable maximal torus of G such that $\dim T_1$ is maximal (see (2.6)), and let B be a Borel subgroup of G containing T such that for any root $\alpha \in R^+(B)$ either $\sigma(\alpha) = \alpha$ or $\sigma(\alpha) \in -R^+(B)$.*

Let E be a T-equivariant nef vector bundle on X of rank n, and let $x \in X$ be a T–fixed point. Then

$$\varepsilon(E, x) = min\{a_i(C)\}_{C,i},$$

where the minimum is taken over all T–invariant curves $C \subset X$ passing through x and integers $\{a_1(C), \cdots, a_n(C)\}$ such that $E\big|_C = \mathcal{O}_C(a_1(C)) \oplus \cdots \oplus \mathcal{O}_C(a_n(C))$.

Proof. We argue as in the proof of Theorem 3.5.

Let W_x denote the exceptional divisor of the blow-up

$$\pi : \widetilde{X} \longrightarrow X$$

at the point $x \in X$. By definition, the Seshadri constant of E at x is the following (see (3.4)):

$$\varepsilon(E, x) = \sup\{\lambda \geq 0 \mid \xi - \lambda q^\star(W_x) \text{ is nef}\}.$$

To prove that $\xi - \lambda q^\star(W_x)$ is nef for a given λ, we need to show that

$$(\xi - \lambda q^\star(W_x)) \cdot D \geq 0,$$

for every curve $D \subset \mathbb{P}(\pi^\star E)$. As before, there exists a T–invariant curve $\widetilde{D} \subset \mathbb{P}(\pi^\star E)$ which is linearly equivalent to D and satisfies the condition that $\widetilde{C} := q(\widetilde{D}) \subset \widetilde{X}$ is a T–invariant curve.

As in the proof of Theorem 3.5, $\xi - \lambda q^\star(W_x)$ is nef if and only if $\mathcal{O}_{\mathbb{P}(\pi^\star E\big|_{\widetilde{C}})}(1) - \lambda q_1^\star(W_x\big|_{\widetilde{C}})$ is nef for every T–invariant curve $\widetilde{C} \subset \widetilde{X}$; see the diagram in (3.6).

We can now complete the proof exactly as done in Theorem 3.5. □

Remark 3.11. Let $X = \overline{G/H}$ be as in Theorem 3.10, and let E be an ample T–equivariant vector bundle on X. Arguing as in Remark 3.7, we have

$$\varepsilon(E, x) \geq 1,$$

for a very general point $x \in X$.

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