Algebra I, Fall 2011
Assignment 4
Due: Monday, Sep 26

(1) Let G be a group. Prove that the relation $a \sim b$ if $b = gag^{-1}$ for some $g \in G$ is an equivalence relation on G.
(2) Does every group of order 35 contain an element of order 5? of order 7?
(3) Does every group whose order is a power of a prime p contain an element of order p?
(4) A finite group G contains an element x of order 10 and also an element y of order 6. What can be said about the order of G?
(5) A group G of order 22 contains elements x and y where $x \neq 1$ and y is not a power of x. Prove that the subgroup generated by these elements is the whole group G.
(6) Prove that every subgroup of index 2 is a normal subgroup. Show by example that a subgroup of index 3 need not be normal.
(7) For which integers n does 2 have a multiplicative inverse in $\mathbb{Z}/\mathbb{Z}n$?
(8) Solve the congruence $2x \equiv 5$ modulo 9 and modulo 6.
(9) *Chinese Reminder Theorem:* Let a, b, u, v be integers and assume that the greatest common divisor of a and b is 1. Then show that there exists an integer x such that $x \equiv u$ modulo a and $x \equiv v$ modulo b. Do the case $u = 0$ and $v = 1$ first.
(10) Prove that the 2-cycle $\sigma = (12)$ and the n-cycle $\tau = (12...n)$ generate the symmetric group S_n.
    Hint:Use conjugation of σ by τ. Also use the result from the last assignment that all 2-cycles generate S_n.
(11) Consider the group P_n of $n \times n$ permutation matrices. That is, P_n consists of all matrices obtained by permuting the rows of identity matrix I_n. We know that P_n is isomorphic to the symmetric group S_n.
    (a) Let $A \in P_n$. Show that the determinant of A is 1 or -1.
    (b) Let $\sigma \in S_n$. Define the *sign* of σ as the determinant of the corresponding permutation matrix. Show that sign of any 2-cycle is -1 and sign of any 3 cycle is 1.
    (c) Consider the map $\phi : S_n \to \{1, -1\}$ given by $\phi(\sigma) = \text{sign}(\sigma)$. Show that ϕ is a group homomorphism. The kernel of ϕ is called the *alternating group* A_n.
    (d) What is the cardinality of A_n?