Algebra I, Fall 2011
Assignment 3
Due: Thursday, Sep 15

(1) Let x and y be elements of a group G. Assume that each of the elements x, y and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G, and that it has order 4.
(2) How many elements of order 2 does the symmetric group S_4 have?
(3) Use row reduction to show that transpositions (i.e. two-cycles) generate the symmetric group S_n, for every $n \geq 1$.
(4) Let $\phi : G \to G'$ be a surjective homomorphism of groups. If G is cyclic, then is it true that G' is cyclic? If G is abelian, then is G' also abelian?
(5) Prove that every group of order less than or equal to 5 is abelian.
(6) Let $\phi : G \to G'$ be an isomorphism of groups. Let $x \in G$. Show that the order of x is the same as the order of $\phi(x)$. What if ϕ is just a homomorphism?
(7) Determine the centre of $GL_n(\mathbb{R})$.
(8) Determine all homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}$. Specify which are injective, which are surjective and which are isomorphisms.
(9) Let G be a group and $a, b \in G$. Show that ab and ba are conjugate.
(10) Let G and G' be two cyclic groups of order n. Show that G and G' are isomorphic.
(11) Let G be a group of order p (p is a prime number). Show that G is isomorphic to a cyclic group. Conclude that there is only one isomorphism class of groups of order p.
(12) Give two groups of order 4 that are not isomorphic. Do the same for groups of order 6.
(13) Find all the automorphisms of a cyclic group of order 10.