Algebra I, Fall 2011
Assignment 2
Due: Wednesday, Sep 7

(1) Let S be a set. Prove that the law of composition defined by $ab = a$ for all $a, b \in S$ is associative. For which sets does this law have an identity?
(2) Let S be a set with an associative law of composition and with an identity. Prove that the subset consisting of the invertible elements in S is a group.
(3) In which of the following cases is H a subgroup of G?
  (a) $G = GL_n(\mathbb{C})$ and $H = GL_n(\mathbb{R})$
  (b) $G = \mathbb{R}^\times$ and $H = \{1, -1\}$.
  (c) $G = GL_2(\mathbb{R})$ and H is the set of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$.
(4) Let G be a group. Define an *opposite group* G^o with law of composition $a \star b$ as follows: the underlying set is the same as G, but the law of composition is $a \star b = ba$. Prove that G^o is a group.
(5) Prove that if a and b are positive integers whose sum is a prime number p, then their greatest common divisor is 1.
(6) Let G be a group and $a, b \in G$. Suppose that a has order 7 and that $a^3 b = b a^3$. Show that $ab = ba$.
(7) An nth root of unity is a complex number z such that $z^n = 1$. Prove that the nth roots of unity form a cyclic subgroup of $\mathbb{C}^\times$ of order n. What is the product of all nth roots of unity?
(8) Let $a, b \in G$ for a group G. Show that ab and ba have the same order.
(9) Determine all groups G that contain no proper subgroup.
(10) True or False? Every subgroup of a cyclic group is cyclic.
(11) Let G be a cyclic group of order 6. How many of its elements generate G? Answer the same question for groups of orders 5, 8 and 9.
(12) Prove that the elementary matrices of the first and third types generate $GL_n(\mathbb{R})$.
(13) True or false? If a, b are elements of finite order in a group G then ab has finite order. What if you assume that G is abelian?
(14) Let H be a subgroup of a group G. We say that H is a *normal* subgroup of G if forevery $a \in H$ and $g \in G$, $gag^{-1} \in H$.
  (a) For a group G, define the *centre* $Z(G)$ of G as follows:
$$Z(G) = \{z \in G | zx = xz \text{ for all } x \in G\}$$
  Show that $Z(G)$ is a normal subgroup of G.
  (b) True or false: Every subgroup of an abelian group is normal.
  (c) True or false: Intersection of normal subgroups is normal.
(15) Let G be cyclic group and let H_1 and H_2 be two subgroups of G. Show that either $H_1 \subset H_2$ or $H_2 \subset H_1$.
(16) True or false: No group can be written as a union of two proper subgroups.
(17) Let G be a cyclic group of order n. Show that for every positive integer m that divides n, there exists exactly one subgroup $H < G$ of order m.
(18) True or false: The group $\mathbb{Q}$ of rational numbers (under addition) is a cyclic group.