Algebra I, Fall 2011
Assignment 1
Due: Wednesday, Aug 24

(1) Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

(2) A square matrix A is called *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible.

(3) (a) Find infinitely many matrices B such that $BA = I_2$ when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$$

(b) Prove that there is no matrix C such that $AC = I_3$.

(4) The *trace* of a square matrix A is the sum of its diagonal entries and is denoted by $\mathrm{tr}(A)$. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Also, if B is invertible, show that $\mathrm{tr}(A) = \mathrm{tr}(BAB^{-1})$.

(5) Show that the equation $AB - BA = I$ has no solutions in $n \times n$ matrices.

(6) How much can a matrix be simplified if **both** row and column operations are allowed?

(7) Prove that every invertible 2×2 matrix can be written as a product of at most four elementary matrices.

(8) Prove that if a product AB of $n \times n$ matrices is invertible then so are the factors A, B.

(9) Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

(10) Consider an arbitrary system of linear equations $AX = B$ where A and B have real entries.

(a) Prove that if $AX = B$ has more than one solution then it has infinitely many.

(b) Prove that if there is a solution in the complex numbers then there is also a real solution.

(11) Prove that the reduced row echelon form obtained by row reduction of a matrix A is uniquely determined by A.

(12) (*Vandermonde determinant*)

(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$

(b) Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column cleverly.

(13) Consider a system of n linear equations in n unknowns: $AX = B$, where A and B have *integer* entries. Prove or disprove the following:

(a) The system has a rational solution if $\det A \neq 0$.

(b) If the system has a rational solution, then it also has an integer solution.

(14) Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

(15) Suppose that $f : M_n \to \mathbb{R}$ is a function satisfying the following conditions. Here M_n refers to the set of all $n \times n$ matrices.

  (i) $f(I) = 1$;
  (ii) f is linear in the rows of a matrix;
  (iii) If two adjacent rows of a matrix A are equal, then $f(A) = 0$.

  Now show that $f(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$.

(16) (a) Suppose that $L_1U_1 = L_2U_2$ where L_i are invertible lower triangular matrices and U_i are invertible upper triangular matrices and all diagonal entries of U_i are 1. Show that $L_1 = L_2$ and $U_1 = U_2$.

  (b) *Most* invertible matrices can be written as a product $A = LU$ where L and U are as above. Part (a) shows that such an expression is unique, if it exists. Explain how to compute L and U when the matrix A is given.

  (c) Give an example of an invertible matrix A which does not have such an expression.