Algebra 4, Homework 6

(1) Show that the correspondence between subgroups and intermediate fields in the main theorem of Galois theory is inclusion reversing.

(2) Let $F \subset L \subset K$ be a tower of fields. Prove or disprove:
   (a) If K/F is Galois, then K/L is Galois.
   (b) If K/F is Galois, then L/F is Galois.
   (c) If K/L and L/F are Galois, then K/F is Galois.
   (d) If $[L : F] = 2$, then L/K is Galois.

(3) Let K be a Galois extension of $\mathbb{Q}$ with Galois group S_3. Is it true that K is the splitting field of an irreducible cubic polynomial over $\mathbb{Q}$?

(4) Let f be a **reducible** quartic polynomial over $\mathbb{Q}$ with **distinct** roots. What are the possible Galois groups of f?

(5) Let K be the splitting field of $x^6 - 25$ over $\mathbb{Q}$. Find the Galois group of K over $\mathbb{Q}$ and determine all the intermediate fields and specify which of them are Galois over $\mathbb{Q}$.

(6) Find the Galois groups of the following polynomials over the indicated fields:
   (a) $(x^2 - 2)(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$, $\mathbb{F}_3$ and $\mathbb{F}_5$.
   (b) $x^4 - 2$ over $\mathbb{Q}$ and $\mathbb{F}_3$.
   (c) $x^4 + 2$ over $\mathbb{Q}$ and $\mathbb{F}_3$.
   (d) $x^4 - 10x^2 + 5$ over $\mathbb{Q}$.
   (e) $x^8 - 2$ over $\mathbb{Q}$.
   (f) $x^3 - 3x - 1$ over $\mathbb{Q}$.
   (g) $x^3 - 10$ over $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-3})$.
   (h) $x^4 - a$ over $\mathbb{Q}$, where a is any integer $\neq 0, \neq \pm 1$ and is squarefree.

(7) Let K be a degree 4 extension of $\mathbb{Q}$ such that it has no intermediate fields. Can K be Galois over $\mathbb{Q}$? Give an example of such a field K.

(8) Given an integer $n \geq 1$, find an example of a Galois extension $F \subset K$ whose Galois group is S_n, the symmetric group on n letters. (consider the field $L(x_1, \ldots, x_n)$ of rational functions over any field L).

   Use this to show that given any finite group G, there is a Galois extension whose Galois group is G.

(9) Given $n \geq 1$, show that there is a field extension $F \subset K$ of degree n with no intermediate fields (use the previous problem).

(10) Find quartic polynomials over $\mathbb{Q}$ whose Galois groups are (a) S_4, (b) D_4, and (c) C_4.

(11) Let $n \geq 1$ be an integer and t an indeterminate. Show that $\mathbb{C}(t)$ is a Galois extension of $\mathbb{C}(t^n + t^{-n})$ and find its Galois group (recall the case $n = 4$ discussed in class).

(12) Let $F = \mathbb{Q}(\sqrt{a})$, where a is a negative integer. Show that F cannot be embedded in a cyclic extension whose degree over $\mathbb{Q}$ is divisible by 4.

(13) Let K/F be a Galois extension with Galois group G and let L be an intermediate field. Let H be the subgroup of G mapping L to itself. Show that H is the normalizer of $\mathrm{Gal}(K/L)$ in G.