Algebra 4, Homework 5

(1) Let $f \in F[x]$ be a polynomial of degree n. Show that the degree of the splitting field of f over F divides $n!$.

(2) Describe the splitting fields of the following polynomials over $\mathbb{Q}$ and find the degree of each.
   (a) $x^2 - 2$
   (b) $x^3 - 2$
   (c) $x^2 + x + 1$
   (d) $x^5 - 7$
   (e) $(x^3 - 2)(x^2 - 2)$
   (f) $x^6 + x^3 + 1$

(3) Let α be a real number such that $\alpha^4 = 5$. Show that
   (a) $\mathbb{Q}(i\alpha^2)$ is normal over $\mathbb{Q}$.
   (b) $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$.
   (c) $\mathbb{Q}(\alpha + i\alpha)$ is not normal over $\mathbb{Q}$.

(4) If the roots of a monic polynomial $f \in F[x]$ in some splitting field are distinct and form a field, then show that char F is a prime p and that $f = x^{p^n} - x$ for some $n \geq 1$.

(5) Let char $f = p$. Let K be a finite extension of F with the degree $[K : F]$ coprime to p. Show that K is separable over F.

(6) Show that the following are equivalent.
   (a) Every algebraic extension of F is separable.
   (b) Either char $F = 0$ or char $F = p$ and every element of F has a p-th root in F.

(7) Given a tower $F \subset L \subset K$ of fields, show that K is separable over F if and only if L is separable over F and K is separable over L.

(8) Let $F \subset K$ be a field extension with the property that every element of K belongs to an intermediate field which is normal over F. Show that K/F is normal.