Algebra 4, Homework 4
Due: Sep 18, 2013

(1) Let $F \subset K \subset L$ be fields. Prove that $\text{tr.deg}_F L = \text{tr.deg}_K L + \text{tr.deg}_F K$.

(2) Let $K = F(\alpha)$ be a field extension generated by a transcendental element α. Let $\beta \in K \setminus F$. Prove that α is algebraic over $F(\beta)$.

(3) Give an example to show that a^b is not necessarily transcendental for two transcendental complex numbers $a, b \in \mathbb{C}$ (transcendental over $\mathbb{Q}$).

(4) Let $\overline{\mathbb{Q}}$ denote an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Suppose that $\alpha_1, \dots, \alpha_n \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$. Is it true that they are algebraically independent?

(5) Let $f \in F[x]$ be an irreducible polynomial of degree over a field F and let K be a quadratic field extension of F. Prove or disprove: either f is irreducible over K, or else f is a product of two irreducible polynomials over K.

(6) (a) Let $p > 2$ be a prime. Prove that exactly half the elements of $\mathbb{F}_p^\times$ are squares.
(b) Prove the above for any finite field of odd order.
(c) Prove that in a finite field of even order, every element is a square.

(7) Find the irreducible polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$. Show that this polynomial is reducible modulo p for every prime p.

(8) Given a finite collection $F_1, \dots, F_n$ of finite fields of the same characteristic, show that there exists a field K which contains subfields isomorphic to each F_i.

(9) Let F, K be fields. Show that K is an algebraic closure of F if and only if K is algebraic over F and for every algebraic extension $F \subset L$, K contains a subfield isomorphic to L.

(10) Let F and L be two fields of the same characteristic which have finite transcendence degrees over the prime field. Show that there exists a field K which contains subfields isomorphic to F and L.